Number of Various Generalized Fibonomial Numbers Not Exceeding a Desired Limit

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Abstract:

The study of generalized Fibonomial numbers, the confluence of Fibonacci numbers and binomial coefficients, has accelerated in the recent time. In this paper we find the number of different generalized Fibonomial numbers not exceeding a desired limit.

1. Introduction:

The sequence of Fibonacci numbers, $\{F_n\}$ is a well-known sequence in Number Theory. The terms of this sequence can be obtained using the recurrence relation $F_n = F_{n-1} + F_{n-2}$; for $n \geq 2$, where $F_0 = 0$ and $F_1 = 1$. Kalman and Mena [3] defined the sequence of generalized Fibonacci numbers as follows:

Definition: For any positive integers a and b, the *generalized Fibonacci numbers* can be obtained using the recurrence relation $F_n^{(a,b)} = aF_{n-1}^{(a,b)} + bF_{n-2}^{(a,b)}$; $n \geq 2$, where $F_0^{(a,b)} = 0$ and $F_1^{(a,b)} = 1$.

First few terms of this sequence are $0, 1, a, a^2 + b, a^3 + 2ab, a^4 + 3a^2b + b^2, \ldots$. Clearly, $F_n^{(1,1)} = F_n$, the traditional n^{th} Fibonacci number, $F_n^{(2,1)} = P_n$, the n^{th} Pell number and $F_n^{(1,2)} = J_n$, the n^{th} Jacobsthal number. The extended Binet formula for $F_n^{(a,b)}$ is given by $F_n^{(a,b)} = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, where $\alpha = \frac{a+\sqrt{a^2+4b}}{2}$ and $\beta = \frac{a-\sqrt{a^2+4b}}{2}$.

Shah and Shah [5] introduced the *generalized Lucas numbers* $L_n^{(a,b)}$ and they defined these numbers as under:

Definition: For any positive integer a and b, the *generalized Lucas numbers* are defined by the recurrence relation $L_n^{(a,b)} = aL_{n-1}^{(a,b)} + bL_{n-2}^{(a,b)}$; $n \geq 2$, where $L_0^{(a,b)} = 2$ and $L_1^{(a,b)} = a$.

First few terms of this sequence are $2, a, a^2 + 2b, a^3 + 3ab, a^4 + a^2b + 4ab + 2b^2, \ldots$. They derived many interesting results related with these numbers and obtained the corresponding extended Binet formula as $L_n^{(a,b)} = \alpha^n + \beta^n$, where $\alpha = \frac{a+\sqrt{a^2+4b}}{2}$ and $\beta = \frac{a-\sqrt{a^2+4b}}{2}$. Clearly $L_n^{(1,1)} = L_n$, the traditional n^{th} Lucas number, $L_n^{(2,1)} = Q_n$, the n^{th} Pell-Lucas number and $L_n^{(1,2)} = j_n$, the n^{th} Jacobsthal-Lucas number. From the extended Binet formulae of generalized Fibonacci numbers and generalized Lucas numbers, it is easy to observe that $F_n^{(a,b)} \leq L_n^{(a,b)}$.

The confluence of binomial coefficients and Fibonacci numbers, namely Fibonomial numbers, is a widely studied topic for a quite long time now. Its generalization has also interested many enthusiastic. Shah and Shah have studied various generalization of Fibonomial numbers in [6], [7] and [8].

1.1 Genomial numbers:

Shah and Shah defined the genorial numbers $F_n^{(a,b)*}$ and genomial numbers $\binom{n}{k}_G$ in [6] as under and obtained interesting properties related to them.

Definition: The *genorial numbers* are defined by

$$F_n^{(a,b)*} = F_n^{(a,b)} \times F_{n-1}^{(a,b)} \times \cdots \times F_2^{(a,b)} \times F_1^{(a,b)} \; ; n > 0 \text{ and } F_0^{(a,b)*} = 1.$$

Definition: The *genomial numbers* are defined by

$$\binom{n}{k}_G = \frac{F_n^{(a,b)*}}{F_k^{(a,b)*} \times F_{n-k}^{(a,b)*}} ; 0 \leq k \leq n.$$

Tables 1 and 2 give first few terms of genorial numbers and genomial numbers respectively.

'Table 1 about here'

'Table 2 about here'

1.2 Double Fibonomial numbers:

In [7], Shah and Shah have introduced double Fibonomial numbers as under and obtained interesting properties of them.

Definition: For any non-negative integer n, *double Fibonorial numbers* are defined as

$$n!!_F \equiv \begin{cases} 1 & ; n = 0 \\ F_n \times F_{(n-2)} \times \cdots \times F_6 \times F_4 \times F_2\, ; & n > 0 \text{ is even} \\ F_n \times F_{(n-2)} \times \cdots \times F_5 \times F_3 \times F_1\, ; & n > 0 \text{ is odd} \end{cases}$$

Definition: For non-negative integers n and k, such that $k \le n$, *double Fibonomial numbers* are defined as

$$\left(\binom{n}{k}\right)_F = \frac{n!!_F}{k!!_F \times (n-k)!!_F}.$$

Tables 3 and 4 displays first few terms of double Fibonorial numbers and double Fibonomial numbers respectively.

'Table 3 about here'

'Table 4 about here'

1.3 Super Fibonomial numbers:

Shah and Shah have defined the super Fibonorial numbers and super Fibonomial numbers in [8] as under:

Definition: For any non-negative integer n, *super Fibonorial numbers* are defined as

$$n!_F^* = \begin{cases} 1 & ; n = 0 \\ n!_F \times (n-1)!_F \times \cdots \times 1!_F ; & n > 0 \end{cases}.$$

It is easy to observe that $n!_F^* = F_1^n \times F_2^{n-1} \times \cdots \times F_n$, for $n > 0$.

Definition: For non-negative integers n and k; such that $n \ge k$, *super Fibonomial numbers* are defined as

$$\left[\!\left[\begin{matrix} n \\ k \end{matrix} \right]\!\right]_F = \frac{n!_F^*}{k!_F^*\,(n-k)!_F^*} = (n-k)!_F \binom{n}{k}_F \binom{n-1}{k-1}_F \cdots \binom{n-k+1}{1}_F.$$

Tables 5 and 6 presents first few terms of super Fibonomial numbers and super Fibonomial numbers respectively. Note that super Fibonomial numbers also satisfies the symmetrical property $\left[\!\left[\begin{matrix} n \\ n-k \end{matrix} \right]\!\right]_F = \left[\!\left[\begin{matrix} n \\ k \end{matrix} \right]\!\right]_F$. Thus, in table 6, we consider only left-side of the Pascal-like triangle for super Fibonomial numbers.

'Table 5 about here'

'Table 6 about here'

In the next section, we find the total number of above mentioned generalized Fibonomial numbers not exceeding any given positive real number Z.

2. Number of genomial numbers not exceeding a desire limit:

Genomial numbers presented in the table 2 demonstrates the structure similar to the Pascal's triangle. This triangle extends down indefinitely. As shown, the array is bordered by 1's. It is symmetric about the vertical line through the middle. In this section we obtain the number of genomial numbers not exceeding any large positive real number Z. We first discard all trivial genomial numbers from the table 2 by cancelling all the bordered 1's and all the numbers on one side of the vertical line through the middle. Thus, we are left with the numbers as shown in the table 7.

'Table 7 about here'

We observe that the first 'cross-row' from the top of the triangle in table 7 gives the sequence $S_1(n)$ of numbers of the form $\binom{n}{1}_G$, so that $S_1(n) = \{S_{1,n}\}_{n\geq 1} = \{a, a^2+b, a^3+2ab, a^4+3a^2b+b^2, a^5+4a^3b+3ab^2, \ldots\}$. The second cross-row gives the sequence $S_2(n)$ of numbers of the form $\binom{n}{2}_G$, so that $S_2(n) = \{S_{2,n}\}_{n\geq 1} = \{a^4+3a^2b+2b^2, a^6+5a^4b+7a^2b^2+2b^3, a^8+7a^6b+16a^4b^2+13a^2b^3+3b^4, \ldots\}$. In general, the r^{th} cross-row consists of sequence $S_r(n) = \{S_{r,n}\}_{n\geq 1}$ of numbers of the form $\binom{n}{r}_G$.

In the following lemma, we show that all these sequences are strictly increasing.

Lemma 2.1: $S_r(n)$ is the strictly increasing sequence for every fixed $r \geq 1$.

Proof: For the generalized Fibonacci numbers, it is easy to observe that $F_{r+n}^{(a,b)} < F_{2r+n}^{(a,b)}$. Therefore,

$$F_{2r+n-1}^{(a,b)} \times F_{2r+n-2}^{(a,b)} \times \cdots \times F_{r+n}^{(a,b)} < F_{2r+n}^{(a,b)} \times F_{2r+n-1}^{(a,b)} \times \cdots \times F_{r+n+1}^{(a,b)}.$$

Further on multiplying $F_{r+n-1}^{(a,b)} \times F_{r+n-2}^{(a,b)} \times \cdots \times F_1^{(a,b)}$ in the numerator and denominator on left hand side and multiplying $F_{r+n}^{(a,b)} \times F_{r+n-1}^{(a,b)} \times \cdots \times F_1^{(a,b)}$ in the numerator and denominator on right hand side, we obtain $\binom{2r+n-1}{r}_G < \binom{2r+n}{r}_G$, that is, $S_{r,n} < S_{r,n+1}$. Hence, $S_r(n)$ is a strictly increasing sequence for all the values of $r \geq 1$, as stated.

To decide the range of r, we first show that the terms of the type $\{S_r(1)\}_{r\geq 1} = \left\{\binom{2r}{r}_G\right\}$ make a strictly increasing sequence.

Lemma 2.2: $\left\{\binom{2r}{r}_G\right\}$ is strictly increasing for every $r \geq 1$.

Proof: By [4], it is known that $F_n^{(a,b)} \leq L_n^{(a,b)}$ and $F_{2n}^{(a,b)} = F_n^{(a,b)} L_n^{(a,b)}$. Thus,

$$\left[F_{r+1}^{(a,b)}\right]^2 \leq F_{2r+2}^{(a,b)} < F_{2r+2}^{(a,b)} F_{2r+1}^{(a,b)}.$$

Using the definition of genorial number, clearly $\frac{F_{2r}^{(a,b)*}}{\left[F_r^{(a,b)*}\right]^2} < \frac{F_{2r+2}^{(a,b)*}}{\left[F_{r+1}^{(a,b)*}\right]^2}$. Also, using the definition of genomial number, we get $\binom{2r}{r}_G < \binom{2r+2}{r+1}_G$. Hence $\left\{\binom{2r}{r}_G\right\}$ is strictly increasing for every $r \geq 1$.

The following theorem gives the asymptotic value of the number of genomial numbers up to a desired limit.

Theorem 2.3: The total number of genomial numbers not exceeding any large positive real number Z is given by $\frac{\log Z}{2\log\alpha}\log\left(\frac{\log Z}{\log\alpha}\right) + \left(C - \frac{1}{2}\right)\frac{\log Z}{\log\alpha} + O\left(\sqrt{\log Z}\right)$, where C is a Euler's constant.

Proof: We first find the number of genomial numbers not exceeding Z in each of the sequences $\{S_{r,n}\}_{n\geq 1}$. For that we need to find t such that $S_{r,t} \leq Z < S_{r,t+1}$. This will be equivalent to $\binom{t+2r-1}{r}_G \leq Z < \binom{t+2r}{r}_G$. Using the definition of $\binom{m}{k}_G$, we get

$$F_{t+2r-1}^{(a,b)} \times F_{t+2r-2}^{(a,b)} \times \cdots \times F_{t+r}^{(a,b)} \leq Z \times F_r^{(a,b)*} < F_{t+2r}^{(a,b)} \times F_{t+2r-1}^{(a,b)} \times \cdots \times F_{t+r+1}^{(a,b)}.$$

By [4], it is known that $\alpha^{n-2} < F_n^{(a,b)} < \alpha^{n-1}$, where $n \geq 3$ and $\alpha = \frac{a+\sqrt{a^2+4b}}{2}$. Thus, if we let $t + 2r = m$, then we get

$$\alpha^{m-3} \times \alpha^{m-4} \times \cdots \times \alpha^{m-r-2} < Z \times F_r^{(a,b)*} < \alpha^{m-1} \times \alpha^{m-2} \times \cdots \times \alpha^{m-r}.$$

This is equivalent to $\alpha^{\frac{r(2m-r-5)}{2}} < Z \times F_r^{(a,b)*} < \alpha^{\frac{r(2m-r-1)}{2}}$. Now we substitute $m = t + 2r$ back so as to get $\alpha^{\frac{r(3r+2t-5)}{2}} < Z \times F_r^{(a,b)*} < \alpha^{\frac{r(3r+2t-1)}{2}}$. This gives

$$\frac{r}{2}(3r + 2t - 5)\log\alpha < \log\left(Z \times F_r^{(a,b)*}\right) < \frac{r}{2}(3r + 2t - 1)\log\alpha.$$

Thus, $t < \frac{\log\left(Z\times F_r^{(a,b)*}\right)}{r\log\alpha} - \frac{(3r-5)}{2} < t + 2$, so that

$$t = \left[\frac{\log\left(Z\times F_r^{(a,b)*}\right)}{r\log\alpha} - \frac{(3r-3)}{2}\right] \text{ or } t = \left[\frac{\log\left(Z\times F_r^{(a,b)*}\right)}{r\log\alpha} - \frac{(3r-1)}{2}\right].$$

Now since $F_n^{(a,b)} = \frac{\alpha^n-\beta^n}{\alpha-\beta}$, we have $F_n^{(a,b)} = \alpha^n\left\{\frac{1-\left(\frac{\beta}{\alpha}\right)^n}{\alpha-\beta}\right\}$, where $\frac{\beta}{\alpha} < 1$ and $\alpha - \beta > 1$. Thus, $\log F_n^{(a,b)} \approx n\log\alpha$. Also, since $\log\left(Z \times F_r^{(a,b)*}\right) = \log Z + \sum_{i=1}^{r}\log F_i^{(a,b)}$, we have $\log\left(Z \times F_r^{(a,b)*}\right) \approx \log Z + \frac{r(r+1)}{2}\log\alpha$. This gives

$$t = \left[\frac{\log Z}{r\log\alpha} - r + 2\right] \text{ or } = \left[\frac{\log Z}{r\log\alpha} - r + 1\right].$$

Therefore, $t \in \left\{\frac{\log Z}{r\log\alpha} - r + 2 + \theta', \frac{\log Z}{r\log\alpha} - r + 1 + \theta'\right\};\ 0 \leq \theta' < 1$. These values can be collaborated as

$$t = \frac{\log Z}{r\log\alpha} - r + 1 + \theta;\ 0 \leq \theta < 2.$$

Here, the value of t gives the number of genomial numbers not exceeding Z in each of the sequence $S_r(n)$, for any fixed r.

To find the total number of genomial numbers not exceeding Z, we next need to find the total number of such sequences. That is, we need to find the value of r such that $\binom{2r}{r}_G \leq Z < \binom{2r+2}{r+1}_G$. Using the definition of genomial numbers, we have

$$\frac{F_{2r}^{(a,b)}\times F_{2r-1}^{(a,b)}\times\cdots\times F_{r+1}^{(a,b)}}{F_r^{(a,b)}\times F_{r-1}^{(a,b)}\times\cdots\times F_1^{(a,b)}} \leq Z < \frac{F_{2r+1}^{(a,b)}\times F_{2r}^{(a,b)}\times\cdots\times F_{r+2}^{(a,b)}}{F_{r+1}^{(a,b)}\times F_r^{(a,b)}\times\cdots\times F_1^{(a,b)}}.$$

Thus, $\frac{\alpha^{2r-2}\times\alpha^{2r-3}\times\cdots\times\alpha^{r-1}}{\alpha^{r-1}\times\alpha^{r-2}\times\cdots\times\alpha^0} < Z < \frac{\alpha^{2r+1}\times\alpha^{2r}\times\cdots\times\alpha^{r+1}}{\alpha^{r-1}\times\alpha^{r-2}\times\cdots\times\alpha^{-1}}$, which can be further simplified to $\alpha^{r(r+1)} < Z < \alpha^{(r+1)(r+2)}$. Therefore $r^2 + r < \frac{\log Z}{\log\alpha} < r^2 + 3r + 2$, which can be rewritten as $r^2 - 2r + 1 < r^2 + r < \frac{\log Z}{\log\alpha} < r^2 + 3r + 2 < r^2 + 4r + 4$. Therefore,

$$(r-1)^2 < \frac{\log Z}{\log\alpha} < (r+2)^2, \text{ that is } r - 1 < \sqrt{\frac{\log Z}{\log\alpha}} < r + 2.$$

Thus, $r < \sqrt{\frac{\log Z}{\log \alpha}} + 1 < r + 3$, which gives $r = \left[\sqrt{\frac{\log Z}{\log \alpha}}\right]$ or $r = \left[\sqrt{\frac{\log Z}{\log \alpha}} - 1\right]$. That is $r \in \left\{\sqrt{\frac{\log Z}{\log \alpha}} + \theta, \sqrt{\frac{\log Z}{\log \alpha}} - 1 + \theta\right\}; 0 \le \theta < 1$. After collaborating these values, we have $r = \sqrt{\frac{\log Z}{\log \alpha}} + \theta^{'}; -1 \le \theta^{'} < 1$.

Hence, the total number of genomial numbers not exceeding a given positive real number Z is

$$\sum_{r=1}^{\sqrt{\frac{\log Z}{\log \alpha}}+\theta^{'}} \left[\frac{\log Z}{r \log \alpha} - r + 1 + \theta\right]; \text{ where } 0 \le \theta < 2 \text{ and } -1 \le \theta^{'} < 1.$$

Considering $\sqrt{\frac{\log Z}{\log \alpha}} + \theta^{'} = \gamma$, above can be written as

$$\left[\frac{\log Z}{\log \alpha} \sum_{r \le \gamma} \frac{1}{r}\right] - \left[\sum_{r \le \gamma} r\right] + \left[(1+\theta) \sum_{r \le \gamma} 1\right].$$

This can be expanded using results from [1] as follows:

$$\frac{\log Z}{\log \alpha}\left\{\log \gamma + C + O\left(\frac{1}{\gamma}\right)\right\} - \left\{\frac{\gamma^2}{2} + O(\gamma)\right\} + (1+\theta)\gamma + O(1).$$

This is same as $\frac{\log Z}{\log \alpha}\left\{\log \sqrt{\frac{\log Z}{\log \alpha}} + C + O\left(\sqrt{\frac{\log \alpha}{\log Z}}\right)\right\} - \left\{\frac{\log Z}{2 \log \alpha} + O\left(\sqrt{\frac{\log Z}{\log \alpha}}\right)\right\}$. Thus, the total number of genomial numbers not exceeding a given positive real number Z is asymptotically equal to $\frac{\log Z}{2 \log \alpha} \log\left(\frac{\log Z}{\log \alpha}\right) + \left(C - \frac{1}{2}\right)\frac{\log Z}{\log \alpha} + O(\sqrt{\log Z})$.

The following corollary states that natural density of genomial numbers in $[1, Z]$ approaches to zero, where Z is any positive real number.

Corollary 2.4: The natural density of genomial numbers approaches to zero.

Proof: From last theorem, the natural density d of genomial numbers is given by

$$d = \lim_{Z \to \infty} \frac{\frac{\log Z}{2 \log \alpha} \log\left(\frac{\log Z}{\log \alpha}\right) + \left(C - \frac{1}{2}\right)\frac{\log Z}{\log \alpha} + O(\sqrt{\log Z})}{Z}.$$

This is same as $d = \lim_{Z \to \infty} \frac{\frac{\log Z}{2 \log \alpha} \log\left(\frac{\log Z}{\log \alpha}\right) + \left(C - \frac{1}{2}\right)\frac{\log Z}{\log \alpha}}{Z} + \lim_{Z \to \infty} \frac{O(\sqrt{\log Z})}{Z}$. Since $Z > \sqrt{\log Z}$, $\lim_{Z \to \infty} \frac{O(\sqrt{\log Z})}{Z}$ vanishes. Therefore, $d = \lim_{Z \to \infty} \frac{\frac{\log Z}{2 \log \alpha} \log\left(\frac{\log Z}{\log \alpha}\right) + \left(C - \frac{1}{2}\right)\frac{\log Z}{\log \alpha}}{Z}$ approaches to zero.

3. Number of double Fibonomial numbers not exceeding a desired limit:

To find the number of double Fibonomial numbers not exceeding some large positive real number Z, here too we discard all the trivial double Fibonomial numbers from the table 4. We are thus left over with the following non-trivial double Fibonomial numbers.

'Table 8 about here'

Here we note that any arbitrary r^{th} cross-row consists of numbers of the type $\left(\binom{n}{r}\right)_F$, which we define to be the sequence $S_r(n) = \{S_{r,n}\}_{n \ge 1}$. We show that these sequences are strictly increasing for every $r \ge 1$.

Lemma 3.1: For the sequence $S_1(n) = \{S_{1,n}\}_{n \ge 1}$,

(a) the subsequence $\{S_{1,1}, S_{1,3}, S_{1,5}, \cdots\}$ is strictly increasing

(b) the subsequence $\{S_{1,2}, S_{1,4}, S_{1,6}, \cdots\}$ is strictly increasing

(c) every term with odd suffix is greater than it's both the neighboring terms with even suffices.

Proof: Clearly, the sequence $S_1(n) = \{S_{1,n}\}_{n\geq 1}$ will form two subsequences $\{S_{1,1}, S_{1,3}, \cdots\}$ and $\{S_{1,2}, S_{1,4}, \cdots\}$. We only prove the results (a) and (c).

We start with $\{S_{1,1}, S_{1,3}, \cdots\} = \left\{\left(\binom{2}{1}\right)_F, \left(\binom{4}{1}\right)_F, \cdots\right\}$, which consists of the elements of the type $\left(\binom{2m}{1}\right)_F = \frac{F_{2m}\times F_{2m-2}\times\cdots\times F_4\times F_2}{F_{2m-1}\times F_{2m-3}\times\cdots\times F_3\times F_1}$; for $m = 1,2,\cdots$. Since $\left(\frac{F_{2m+2}}{F_{2m+1}}\right) > 1$, we get $S_{1,2m+1} > S_{1,2m-1}$. This shows that the subsequence considered is strictly increasing.

Further (c) follows easily from the known result for the Fibonacci numbers that $F_{2n}^2 < F_{2n-1}F_{2n+1}$.

Lemma 3.2: The sequence $S_r(m) = \{S_{r,1}, S_{r,2}, \ldots\}$; $r > 1$ is strictly increasing.

Proof: For the fixed r, we show that $S_{r,t} < S_{r,t+1}$; for all t, that is $\frac{t!!_F}{(t-r)!!_F} < \frac{(t+1)!!_F}{(t-r+1)!!_F}$. Now for the even value of r, this inequality reduces to

$$F_t \times F_{t-2} \times \cdots \times F_{t-r+1} < F_{t+1} \times F_{t-1} \times \cdots \times F_{t-r+2},$$

which always holds since each term on right side is bigger than the corresponding term of left side. Also, for odd value of r, we prove the result only when t is even. The other case can be proved accordingly. Now when t is even, the required inequality reduces to

$$\frac{F_{t-r+1}\times F_{t-r-1}\times\cdots\times F_2}{F_{t-r}\times F_{t-r-2}\times\cdots\times F_1} < \frac{F_{t+1}\times F_{t-1}\times\cdots\times F_3\times F_1}{F_t\times F_{t-2}\times\cdots\times F_2}.$$

Here both the sides of this inequality consist of ratios of consecutive Fibonacci numbers. Since $\frac{F_{2n+1}}{F_{2n}} \geq \frac{F_{2n}}{F_{2n-1}}$ and the number of such ratios on the right side is greater than that of left side, the required inequality follows. This proves the required result.

In the following lemma, we show that the terms of the right most column of the table 8 also form a strictly increasing sequence.

Lemma 3.3: $\left(\binom{2r}{r}\right)_F$ is strictly increasing sequence for every $r \geq 1$.

Proof: To prove the result, it is enough to show that $F_{2r+2}r!!_F > (r+1)!!_F$. As for any value of r, $F_{2r+2} > F_{r+1}$ is always true, the result follows.

The following theorem now gives the number of double Fibonomial numbers up to the desired limit.

Theorem 3.4: The number of double Fibonomial numbers not exceeding any large positive real number Z is asymptotically equal to

$$\frac{\log Z}{\log\alpha}\log\left(\sqrt{\frac{\log Z}{2\log\alpha}} - \chi_1\right) + \left(C - \frac{1}{4}\right)\frac{\log Z}{\log\alpha} + \chi_2 + O\left(\sqrt{\log Z}\right);$$

where $\chi_1 = \begin{cases} 0; r \text{ is even} \\ \frac{1}{2}; r \text{ is odd} \end{cases}$ and $\chi_2 = \begin{cases} 0 & ; r \text{ is even} \\ \sqrt{\frac{\log Z}{8\log\alpha}} - \frac{1}{8}; r \text{ is odd} \end{cases}$.

Proof: We first find the number of generalized double Fibonomial numbers not exceeding Z in the sequence $S_r(n) = \{S_{r,n}\}_{n\geq 1}$. For that we need to find t such that $S_{r,t} \leq Z < S_{r,t+1}$, which is equivalent to

$\frac{(t+2r-1)!!_F}{(t+r-1)!!_F} \le Z \times r!!_F < \frac{(t+2r)!!_F}{(t+r)!!_F}$. Now depending on the parity of t and r, we consider the four cases. When both t and r are even positive integers, we get

$$F_{t+2r-1} \times F_{t+2r-3} \times \cdots \times F_{t+r+1} \le Z \times r!!_F < F_{t+2r} \times F_{t+2r-2} \times \cdots \times F_{t+r+2}.$$

Therefore, we have

$$\alpha^{(t+2r-3)+(t+2r-5)+\cdots+(t+r-1)} \le Z \times r!!_F < \alpha^{(t+2r-1)+(t+2r-3)+\cdots+(t+r+1)}.$$

This is equivalent to $\alpha^{\frac{r(2t+3r-4)}{4}} \le Z \times r!!_F < \alpha^{\frac{r(2t+3r)}{4}}$, which is same as

$$t \le \frac{1}{2}\left(\frac{4\log(Z \times r!!_F)}{r\log\alpha} - 3r + 4\right) < t + 2.$$

Thus, we get either $t = \left[\frac{2\log(Z \times r!!_F)}{r\log\alpha} - \frac{3}{2}r + 2\right]$ or $t = \left[\frac{2\log(Z \times r!!_F)}{r\log\alpha} - \frac{3}{2}r + 1\right]$. For the other three cases too, we get the similar values for t. Now, when r is even, $\log(Z \times r!!_F) = \log Z + (\log F_2 + \log F_4 + \cdots + \log F_r)$. Since $\log F_n = n\log\alpha$; where $\alpha = \frac{1+\sqrt{5}}{2}$, we get $\log(Z \times r!!_F) = \log Z + (2\log\alpha + 4\log\alpha + \cdots + r\log\alpha) = \log Z + \frac{r(r+2)}{4}\log\alpha$. Thus, either $t = \left[\frac{2\log Z}{r\log\alpha} - r + 3\right]$ or $t = \left[\frac{2\log Z}{r\log\alpha} - r + 2\right]$. This can be written as $t \in \left\{\frac{2\log Z}{r\log\alpha} - r + 3 + \theta', \frac{2\log Z}{r\log\alpha} - r + 2 + \theta'\right\}$; $0 \le \theta' < 1$. These values can be collaborated as

$$t = \frac{2\log Z}{r\log\alpha} - r + 2 + \theta;\ 0 \le \theta < 2.$$

Here, the value of t gives the number of double Fibonomial numbers not exceeding Z in each of the sequence $S_r(n)$, for any fixed even integer r. To find the required number of double Fibonomial numbers, we take the summation over r. Again, to find the upper limit of this sum we need to find the value of r, such that $\left(\binom{2r}{r}\right)_F \le Z < \left(\binom{2r+2}{r+1}\right)_F$, where $\left(\binom{2r}{r}\right)_F$ represents right most column in table 8. This inequality can be rewritten as

$$\frac{F_{2r} \times F_{2r-2} \times \cdots \times F_2}{(F_r \times F_{r-2} \times \cdots \times F_1)^2} \le Z < \frac{F_{2r+2} \times F_{2r} \times \cdots \times F_2}{(F_{r+1} \times F_{r-1} \times \cdots \times F_2)^2},$$

which is equivalent to $\frac{\alpha^{2r-2} \times \alpha^{2r-4} \times \cdots \times \alpha^0}{(\alpha^{r-1} \times \alpha^{r-3} \times \cdots \times \alpha^0)^2} \le Z < \frac{\alpha^{2r+1} \times \alpha^{2r-1} \times \cdots \times \alpha^1}{(\alpha^{r-1} \times \alpha^{r-3} \times \cdots \times \alpha^0)^2}$. This gives $\alpha^{\frac{(r-1)^2}{2}} \le Z < \alpha^{\frac{(r+2)^2-1}{2}} < \alpha^{\frac{(r+2)^2}{2}}$, that is $r \le \sqrt{\frac{2\log Z}{\log\alpha}} + 1 < r + 3$. Thus, we have $r = \left[\sqrt{\frac{2\log Z}{\log\alpha}}\right]$ or $r = \left[\sqrt{\frac{2\log Z}{\log\alpha}}\right]$ or $r = \left[\sqrt{\frac{2\log Z}{\log\alpha}} - 1\right]$. That is, $r \in \left\{\sqrt{\frac{2\log Z}{\log\alpha}} + 1 + \theta, \sqrt{\frac{2\log Z}{\log\alpha}} + \theta, \sqrt{\frac{2\log Z}{\log\alpha}} - 1 + \theta\right\}; 0 \le \theta < 1$. After collaborating these values, we have $r = \sqrt{\frac{2\log Z}{\log\alpha}} + \theta'; -1 \le \theta' < 2$. Hence, total number of double Fibonomial numbers not exceeding a positive real number Z is $\sum_{r=1}^{\sqrt{\frac{2\log Z}{\log\alpha}}+\theta'} \left\{\frac{2\log Z}{r\log\alpha} - r + 2 + \theta\right\}$; where $0 \le \theta < 2$, $-1 \le \theta' < 2$ and r is an even integer. Considering $\sqrt{\frac{2\log Z}{\log\alpha}} + \theta' = \gamma$ and $r = 2m$, this sum can be written as

$$\left[\frac{\log Z}{\log\alpha}\sum_{m \le \frac{\gamma}{2}} \frac{1}{m}\right] - 2\left[\sum_{m \le \frac{\gamma}{2}} m\right] + \left[(2+\theta)\sum_{m \le \frac{\gamma}{2}} 1\right].$$

Using results from [1], this can be expanded as

$$\frac{\log Z}{\log\alpha}\left\{\log\left(\frac{\gamma}{2}\right)+C+O\left(\frac{2}{\gamma}\right)\right\}-\left\{\frac{\gamma^2}{8}+O\left(\frac{\gamma}{2}\right)\right\}+\frac{(2+\theta)\gamma}{2}+O(1).$$

This is same as $\frac{\log Z}{\log\alpha}\left\{\log\left(\sqrt{\frac{\log Z}{2\log\alpha}}\right)+C+O\left(\sqrt{\frac{2\log\alpha}{\log Z}}\right)\right\}-\left\{\frac{\log Z}{4\log\alpha}+O\left(\sqrt{\frac{\log Z}{2\log\alpha}}\right)\right\}$. Thus, the total number of double Fibonomial numbers not exceeding a given positive real number Z, when r is an even integer, is

$$\frac{\log Z}{\log\alpha}\log\left(\sqrt{\frac{\log Z}{2\log\alpha}}\right)+\left(C-\frac{1}{4}\right)\frac{\log Z}{\log\alpha}+O\left(\sqrt{\log Z}\right).$$

Using the similar technique, the total number of double Fibonomial numbers not exceeding a given positive real number Z, when r is an odd integer, can be obtained as

$$\frac{\log Z}{\log\alpha}\log\left(\sqrt{\frac{\log Z}{2\log\alpha}}-\frac{1}{2}\right)+\left(C-\frac{1}{4}\right)\frac{\log Z}{\log\alpha}+\left(\sqrt{\frac{\log Z}{8\log\alpha}}-\frac{1}{8}\right)+O\left(\sqrt{\log Z}\right).$$

If we let $\chi_1=\begin{cases}0; r \text{ is even}\\ \frac{1}{2}; r \text{ is odd}\end{cases}$ and $\chi_2=\begin{cases}0 & ; r \text{ is even}\\ \sqrt{\frac{\log Z}{8\log\alpha}}-\frac{1}{8}; & r \text{ is odd}\end{cases}$, then combining the result for all the positive values of r, we get the total number of double Fibonomial numbers not exceeding a given positive real number Z as

$$\frac{\log Z}{\log\alpha}\log\left(\sqrt{\frac{\log Z}{2\log\alpha}}-\chi_1\right)+\left(C-\frac{1}{4}\right)\frac{\log Z}{\log\alpha}+\chi_2+O\left(\sqrt{\log Z}\right).$$

The following corollary states the natural density of double Fibonomial numbers approaches to zero.

Corollary 3.5: The natural density of double Fibonomial numbers approaches to zero.

Proof: Using the above theorem, the natural density d of double Fibonomial numbers is given by

$$d=\lim_{Z\to\infty}\frac{\frac{\log Z}{\log\alpha}\log\left(\sqrt{\frac{\log Z}{2\log\alpha}}-\chi_1\right)+\left(C-\frac{1}{4}\right)\frac{\log Z}{\log\alpha}+\chi_2+O\left(\sqrt{\log Z}\right)}{Z}.$$

This is same as $d=\lim_{Z\to\infty}\frac{\frac{\log Z}{\log\alpha}\log\left(\sqrt{\frac{\log Z}{2\log\alpha}}-\chi_1\right)+\left(C-\frac{1}{4}\right)\frac{\log Z}{\log\alpha}+\chi_2}{Z}+\lim_{Z\to\infty}\frac{O\left(\sqrt{\log Z}\right)}{Z}$. Since $Z>\sqrt{\log Z}$ $\lim_{Z\to\infty}\frac{O\left(\sqrt{\log Z}\right)}{Z}$ vanish. Therefore, $d=\lim_{Z\to\infty}\frac{\frac{\log Z}{\log\alpha}\log\left(\sqrt{\frac{\log Z}{2\log\alpha}}-\chi_1\right)+\left(C-\frac{1}{4}\right)\frac{\log Z}{\log\alpha}+\chi_2}{Z}$ approaches to zero.

Hence, the natural density of double Fibonomial numbers in the set $[1, Z]$ approaches to zero, where Z is any positive real number.

4. Number of super Fibonomial numbers not exceeding a desired limit:

In this section too, to obtain the number of super Fibonomial numbers not exceeding given positive real number Z, we first discard all trivial super Fibonomial numbers from the table 6 to obtain non-trivial super Fibonomial numbers.

‘Table 9 about here’

We observe that the r^{th} cross-row consists of numbers of the type $\left[\!\left[{n \atop r}\right]\!\right]_F$, which we define to be the sequence $S_r(n)=\{S_{r,n}\}_{n\geq 1}$. The following lemma states that $S_r(n)$ is the strictly increasing sequence for fixed $r\geq 1$, which can be easily proved using the definition of super Fibonomial numbers.

Lemma 4.1: $S_r(n)$ is the strictly increasing sequence for fixed $r\geq 1$.

Note that the right most column of table 9 contains the elements of the type $\left[\!\left[\begin{matrix}2r\\ r\end{matrix}\right]\!\right]_F$. To decide the range of r, we need to find r, such that $\left[\!\left[\begin{matrix}2r\\ r\end{matrix}\right]\!\right]_F \leq Z < \left[\!\left[\begin{matrix}2r+2\\ r+1\end{matrix}\right]\!\right]_F$. But we first show that these terms make a strictly increasing sequence, which can again be proved using the definition of super Fibonomial numbers.

Lemma 4.2: $\left[\!\left[\begin{matrix}2r\\ r\end{matrix}\right]\!\right]_F$ is strictly increasing for every $r \geq 1$.

Following theorem gives the number of super Fibonomial numbers up to a desired limit.

Theorem 4.3: The total number of super Fibonomial numbers not exceeding any large positive real number Z is given by $\sum_{r=1}^{\sqrt[3]{\frac{\log Z}{\log\alpha}}+1+\theta'} \left[\sqrt{\frac{2\log Z}{r\log\alpha} + \frac{r^2+4r+9}{4}} - \frac{3}{2}r + \theta\right]$; where $-2 \leq \theta < 2$ and $0 \leq \theta' < 1$, as required.

Proof: We first find the number of super Fibonomial numbers not exceeding Z in each of the sequences $\{S_{r,n}\}_{n\geq 1}$. For that we find t such that $S_{r,t} \leq Z < S_{r,t+1}$, which is equivalent to $\left[\!\left[\begin{matrix}t+2r-1\\ r\end{matrix}\right]\!\right]_F \leq Z < \left[\!\left[\begin{matrix}t+2r\\ r\end{matrix}\right]\!\right]_F$. This gives $\frac{(t+2r-1)!_F^*}{(t+r-1)!_F^*} \leq Z \times r!_F^* < \frac{(t+2r)!_F^*}{(t+r)!_F^*}$. Again, since $\alpha^{(n-2)} \leq F_n \leq \alpha^{(n-1)}$, we get $\alpha^{\frac{n(n-3)}{2}} \leq n!_F \leq \alpha^{\frac{n(n-1)}{2}}$ and consequently we get

$$\alpha^{\frac{n(n-4)(n+1)}{6}} \leq n!_F^* \leq \alpha^{\frac{n(n-1)(n+1)}{6}}.$$

Therefore, we have $\frac{\alpha^{\frac{(t+2r-1)(t+2r-5)(t+2r)}{6}}}{\alpha^{\frac{(t+r-1)(t+r-5)(t+r)}{6}}} \leq Z \times r!_F^* < \frac{\alpha^{\frac{(t+2r-1)(t+2r+1)(t+2r)}{6}}}{\alpha^{\frac{(t+r-1)(t+r+1)(t+r)}{6}}}$. This can be further simplified as $\alpha^{\frac{r(7r^2+3t^2+9tr-18r-12t+5)}{6}} \leq Z \times r!_F^* < \alpha^{\frac{r(7r^2+3t^2+9tr-1)}{6}}$, which is equivalent to $7r^2 + 3t^2 + 9tr - 18r - 12t + 5 \leq \frac{6\log(Z\times r!_F^*)}{r\log\alpha} < 7r^2 + 3t^2 + 9tr - 1$. Thus,

$$3\left(t+\frac{3}{2}r\right)^2 - 12\left(t+\frac{3}{2}r\right) + 5 \leq \frac{6\log(Z\times r!_F^*)}{r\log\alpha} - \frac{r^2}{4} < 3\left(t+\frac{3}{2}r\right)^2 - 1.$$

From this it can be seen that $t \leq \sqrt{\frac{6\log(Z\times r!_F^*)}{3r\log\alpha} - \frac{r^2}{12} + \frac{7}{3}} - \frac{3}{2}r + 2 < t + 4$. This gives

$$t \in \left\{\begin{matrix}\left[\sqrt{\frac{6\log(Z\times r!_F^*)}{3r\log\alpha} - \frac{r^2}{12} + \frac{7}{3}} - \frac{3}{2}r + 2\right], \left[\sqrt{\frac{6\log(Z\times r!_F^*)}{3r\log\alpha} - \frac{r^2}{12} + \frac{7}{3}} - \frac{3}{2}r + 1\right], \\ \left[\sqrt{\frac{6\log(Z\times r!_F^*)}{3r\log\alpha} - \frac{r^2}{12} + \frac{7}{3}} - \frac{3}{2}r\right], \left[\sqrt{\frac{6\log(Z\times r!_F^*)}{3r\log\alpha} - \frac{r^2}{12} + \frac{7}{3}} - \frac{3}{2}r - 1\right], \\ \left[\sqrt{\frac{6\log(Z\times r!_F^*)}{3r\log\alpha} - \frac{r^2}{12} + \frac{7}{3}} - \frac{3}{2}r - 2\right]\end{matrix}\right\}.$$

Since $\log(Z \times r!_F^*) \approx \log Z + \frac{r(r+1)(r+2)}{6}\log\alpha$, we get

$$t \in \left\{\begin{matrix}\left[\sqrt{\frac{2\log Z}{r\log\alpha} + \frac{r^2+4r+9}{4}} - \frac{3}{2}r + 2\right], \left[\sqrt{\frac{2\log Z}{r\log\alpha} + \frac{r^2+4r+9}{4}} - \frac{3}{2}r + 1\right], \\ \left[\sqrt{\frac{2\log Z}{r\log\alpha} + \frac{r^2+4r+9}{4}} - \frac{3}{2}r\right], \left[\sqrt{\frac{2\log Z}{r\log\alpha} + \frac{r^2+4r+9}{4}} - \frac{3}{2}r - 1\right], \\ \left[\sqrt{\frac{2\log Z}{r\log\alpha} + \frac{r^2+4r+9}{4}} - \frac{3}{2}r - 2\right]\end{matrix}\right\}.$$

Thus, $t \in \left\{ \begin{array}{c} \sqrt{\frac{2\log Z}{r\log\alpha}+\frac{r^2+4r+9}{4}}-\frac{3}{2}r+2+\theta', \sqrt{\frac{2\log Z}{r\log\alpha}+\frac{r^2+4r+9}{4}}-\frac{3}{2}r+1+\theta', \\ \sqrt{\frac{2\log Z}{r\log\alpha}+\frac{r^2+4r+9}{4}}-\frac{3}{2}r+\theta', \sqrt{\frac{2\log Z}{r\log\alpha}+\frac{r^2+4r+9}{4}}-\frac{3}{2}r-1+\theta', \\ \sqrt{\frac{2\log Z}{r\log\alpha}+\frac{r^2+4r+9}{4}}-\frac{3}{2}r-2+\theta' \end{array} \right\}$; where $0 \le \theta' < 1$. These values can be collaborated as

$$t = \sqrt{\frac{2\log Z}{r\log\alpha}+\frac{r^2+4r+9}{4}}-\frac{3}{2}r+\theta; \text{ where } -2 \le \theta < 2.$$

This value gives the number of super Fibonomial numbers not exceeding Z in each of the sequence $S_r(m)$.

Further, we need to find the value of r such that $\left[\!\left[\begin{matrix} 2r \\ r \end{matrix} \right]\!\right]_F \le Z < \left[\!\left[\begin{matrix} 2r+2 \\ r+1 \end{matrix} \right]\!\right]_F$, that is $\alpha^{\frac{r^2(3r-1)}{3}} \le Z < \alpha^{r^3}$. It can be easily observed that $r = \sqrt[3]{\frac{\log Z}{\log\alpha}}+1+\theta'; 0 \le \theta' < 1$. Hence, the total number of super Fibonomial numbers not exceeding a given positive real number Z is given by $\sum_{r=1}^{\sqrt[3]{\frac{\log Z}{\log\alpha}}+1+\theta'} \left[\sqrt{\frac{2\log Z}{r\log\alpha}+\frac{r^2+4r+9}{4}}-\frac{3}{2}r+\theta \right]$; where $-2 \le \theta < 2$ and $0 \le \theta' < 1$, as required.

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Tables:

n	$F_n^{(a,b)*}$
0	1
1	1
2	a
3	$a^3 + ab$
4	$a^6 + 3a^4b + 2a^2b^2$
5	$a^{10} + 6a^8b + 12a^6b^2 + 9a^4b^3 + 2a^2b^4$
6	$a^{15} + 10a^{13}b + 39a^{11}b^2 + 75a^9b^3 + 74a^7b^4 + 35a^5b^5 + 6a^3b^6$

Table 1: Genorial numbers

						1						
					1		1					
				1		a		1				
			1		$a^2 + b$		$a^2 + b$		1			

		1		$a^3 + 2ab$		$a^4 + 3a^2b + 2b^2$		$a^3 + 2ab$		1		
	1		$a^4 + 3a + b^2$		$a^6 + 5a^4b + 7a^2b + 2b^3$		$a^6 + 5a^4b + 7a^2b + 2b^3$		$a^4 + 3a + b^2$		1	
1		$a^5 + 4a^3b + 3ab^2$		$a^8 + 7a^6b + 16a^4b^2 + 13a^2b^3 + 3b^4$		$a^9 + 8a^7b + 22a^5b^2 + 23a^3b^3 + 6ab^4$		$a^8 + 7a^6b + 16a^4b^2 + 13a^2b^3 + 3b^4$		$a^5 + 4a^3b + 3ab^2$		1

Table 2: Genomial numbers

n	0	1	2	3	4	5	6	7	8	9	10
$n!!_F$	1	1	1	2	3	10	24	130	504	4420	27720

Table 3: Double Fibonorial numbers

								1								
							1		1							
						1		1		1						
					1		2		2		1					
				1		$\frac{3}{2}$		3		$\frac{3}{2}$		1				
			1		$\frac{10}{3}$		5		5		$\frac{10}{3}$		1			
		1		$\frac{24}{10}$		8		6		8		$\frac{24}{10}$		1		
	1		$\frac{65}{12}$		13		$\frac{65}{3}$		$\frac{65}{3}$		13		$\frac{65}{12}$		1	
1		$\frac{252}{65}$		21		$\frac{126}{5}$		56		$\frac{126}{5}$		21		$\frac{252}{65}$		1

Table 4: Double Fibonomial numbers

n	$n!_F^*$
0	1
1	1
2	1
3	2
4	12
5	360
6	86400
7	269568000
8	17662095360000

Table 5: Super Fibonorial numbers

								1
							1	
						1		1
					1		2	
				1		6		12
			1		30		180	
		1		240		7200		21600
	1		3120		74880		11232000	
1		65520		204422400		24530688000		122653440000

Table 6: Super Fibonomial numbers

				a
			$a^2 + b$	
		$a^3 + 2ab$		$a^4 + 3a^2b + 2b^2$
	$a^4 + 3a^2b + b^2$		$a^6 + 5a^4b + 7a^2b^2 + 2b^3$	
$a^5 + 4a^3b + 3ab^2$		$a^8 + 7a^6b + 16a^4b^2 + 13a^2b^3 + 3b^4$		$a^9 + 8a^7b + 22a^5b^2 + 23a^3b^3 + 6ab^4$

Table 7: Non-trivial genomial numbers

						1
					2	
				$\frac{3}{2}$		3
			$\frac{10}{3}$		5	
		$\frac{24}{10}$		8		6
	$\frac{65}{12}$		13		$\frac{65}{3}$	
$\frac{252}{65}$		21		$\frac{126}{5}$		56

Table 8: Non-trivial double Fibonomial numbers

						1
					2	
				6		12
			30		180	
		240		7200		2160
	3120		74880		11232000	
65520		204422400		24530688000		122653440000

Table 9: Non-trivial super Fibonomial numbers

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