Solution of the non-linear fractional sir epidemic model using fractional reduced differential transform method

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Abstract:

To solve the non-linear SIR epidemic model of fractional order, an efficient methodology known as the fractional reduced differential transform method (FRDTM) is used in this research paper. In medical research, the fractional SIR epidemic model is a system of fractional ordinary differential equations used to investigate the epidemiology and the treatment of sick patients. The Caputo sense is used to characterize the fractional derivative in this research. We get a power series solution of non-linear SIR epidemic model of fractional order with the fractional reduced differential transform method (FRDTM). Also, in this research paper, we use the fractional ordered non-linear SIR epidemic model to demonstrate the efficacy of the fractional reduced differential transform method (FRDTM) and compared findings with residual power series method (RPSM). The approximate derived solutions are shown in graphical form and quantitatively analyzed. As can be seen from these numbers and graphs, for resolving a broad variety of real-world situations involving fractional differential equations we can trust the fractional reduced differential transform method (FRDTM).

Keywords: Fractional reduced differential transform method (FRDTM); Fractional SIR epidemic model; Caputo concept; System of fractional ordinary differential equations; Epidemic dynamics.

MSC2020 Classification: 26A33, 34A08.

Introduction:

Since the beginning of time, contagious illnesses and epidemics have been a severe danger to the population's safety, causing severe consequences not just for humanity but also for civilization's economy and social development. Several of the most pressing concerns in applied mathematics, human physiology, engineering, physical science, as well as other scientific fields is the development and implementation methods for constructing mathematical models of real-world occurrences. Many contagious diseases have raised our concerns towards them, such as measles, rubella, hepatitis B, hepatitis E, hepatitis C, HIV, poliovirus (Banatvala & Brown, 2004; Grossman et al., 2006; Perry & Halsey, 2004; Poynard et al., 2003; Sarin & Kumar, 2012; Tiollais et al., 1985; Wimmer et al., 1993). We have recently seen how devastatingly the COVID-19 pandemic has affected the whole world (Ciotti et al., 2020). Mathematical models are essential for understanding how the infection spreads and for developing new strategies for halting its spread (Grassly & Fraser, 2008). Kermack et al. (William Ogilvy Kermack, 1927) brought the SIR epidemiology model for the first time in the nineteenth century. There are three groups in a SIR epidemic model: individuals who are susceptible (S), individuals who are infected (I), and the individuals who have recovered (R) (William Ogilvy Kermack, 1927). In this research paper, we pay attention to the following fractional ordered highly non-linear SIR epidemic model having arbitrary order:

$$\begin{cases} {}_{0}^{C}\Delta_{t}^{\alpha}S(t) = -m_1 S(t)I(t), \\ {}_{0}^{C}\Delta_{t}^{\alpha}I(t) = m_1 S(t)I(t) - m_2 I(t), \\ {}_{0}^{C}\Delta_{t}^{\alpha}R(t) = m_2 I(t). \end{cases} \tag{1}$$

The concerned initial conditions are $S(0) = S_0$, $I(0) = I_0$, and $R(0) = R_0$. Also, the order of the fractional derivative Δ in reference to time 't' taken through Caputo sense is $0 < \alpha \le 1$, m_1, and m_2 are constants whose values are always positive, that are referred to as the infection rate and the removal rate, correspondingly. As a result, we would not take into account population turnover (birth or death), as well as all diseases, are presumed to stop with recovery. If we add together all three populations, we get the total size of the population (S + I + R). Also, many studies of epidemic models are examined, along with their histories and potential remedies (Allen, 1994; Becker, 1979; Hethcote, 1994). Many mathematicians have solved the SIR epidemic model using novel mathematical techniques (Hasan et al., 2019; Yildirim & Cherruault, 2009); moreover, many mathematicians have also solved the fractional SIR epidemic model (Ameen & Novati, 2017; Arqub & El-Ajou, 2013; Haq et al., 2017; Hasan et al., 2019; Sene, 2020).

Here we employ the fractional reduced differential transform method (FRDTM) aimed at figuring out a highly non-linear SIR epidemic model of fractional order, which was initially based on the differential transform method (DTM). The differential transform method (DTM) was initially coined and invented in the year 1986 by Zhou (Zhou, 1986), a mathematician from China, to solve the problems of mathematics in electrical circuits. DTM is based on classic Taylor's series method, then in the year 2009, Keskin et al. (Keskin & Oturanç, 2009) developed the reduced differential transform method (RDTM) to solve partial differential equations. Moreover, in the year 2011, Gupta et al. (Gupta, 2011) used the fractional reduced differential transform method (FRDTM); in this case, they were the first who used FRDTM to solve fractional ordered Benney–Lin equation. The fractional reduced differential transform method (FRDTM) seems to be a very robust but effortless mathematical technique that can solve highly non-linear partial differential equations (PDEs) and ordinary differential equations (ODEs), and that is the reason we are using FRDTM in this paper to solve the fractional SIR epidemic model. Many noble mathematicians have employed the fractional reduced differential transform method (FRDTM) to solve various fractional differential equations (Abuteen et al., 2016; Patel & Tandel, 2021; Rawashdeh, 2017; Singh & Kumar, 2018; Singh & Srivastava, 2015; Srivastava et al., 2014; Tamboli & Tandel, 2022; Tandel et al., 2022). In this paper, a comparison is made between the fractional reduced differential transform method's (FRDTM) output and the residual power series method's (RPSM) (Hasan et al., 2019) output graphically for precision and pertinency.

The remaining research paper is laid out in the following manner: Section 2 provides specific findings, fundamental definitions, and the fractional calculus (FC) employed in the research study. Section *0* lays out the fundamentals of the fractional reduced differential transform method (FRDTM) and its attributes. The fractional reduced differential transform method

(FRDTM) is used for the non-linear SIR epidemic model of fractional order under Section *0*. Results, tables, and figures for the suggested highly non-linear SIR epidemic model of fractional order are addressed within Section *0*, that is here the most important section of this research paper.

Fractional Calculus:

This section covers the fundamentals of fractional calculus (FC), as well as its preliminaries and notations. Differentiation and integration have a plethora of definitions in mathematical research papers (Balint et al., 2022; Din et al., 2022; Ebaid & Al-Jeaid, 2022; Lan, 2022; Liang et al., 2022; Ruziev & Zunnunov, 2022; Sene, 2022; Vellappandi et al., 2022). Many research articles show that the Riemann-Liouville (RL), as well as Caputo definitions of fractional calculus (FC), are the most widely used.

Definition 2.1. In the following equations, the fractional ordered (α) derivatives in the Caputo and Riemann-Liouville (RL) sense are presented (Luchko, 2022)*:*

$${}_{0}^{RL}\Delta_t^{\alpha} Y(t) = \frac{d^m}{dt^m}\left(P_t^{n-\alpha} O(t)\right) = \begin{cases} \dfrac{d^n O(t)}{dt^n}, & \alpha = n \in □, \\ \dfrac{1}{\Gamma(n-\alpha)} \dfrac{d^n}{dt^n} \displaystyle\int_0^t \dfrac{Y(\varepsilon)}{(t-\varepsilon)^{t-n+1}} d\varepsilon, & 0 \le n-1 < \alpha < n, \end{cases} \quad (2)$$

$${}_{0}^{RL}\Delta_t^{\alpha} t^n = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} t^{n-\alpha}, n > 0 \quad (3)$$

and,

$${}_{0}^{C}\Delta_\tau^{\sigma} O(t) = P_t^{n-\alpha}\left(\frac{d^n}{dt^n} O(t)\right) = \begin{cases} \dfrac{d^n O(t)}{dt^n}, & \alpha = n \in □, \\ \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t \dfrac{O^m(\varepsilon)}{(t-\varepsilon)^{\alpha-n+1}} d\varepsilon, & 0 \le n-1 < \alpha < n. \end{cases} \quad (4)$$

where $n \in □$ and t denotes the time, whose value is always positive. For $n-1 < \alpha \le n$ and $O \in L_1[a,b]$, these derivatives have two following essential properties:

$$\begin{cases} \left({}_{0}^{C}\Delta_t^{\alpha} P_t^{\alpha} O\right)(t) = O(t) \\ \left(P_t^{\alpha}\, {}_{0}^{C}\Delta_t^{\alpha} O\right)(t) = O(t) - \displaystyle\sum_{r=0}^{n-1} O^r\left(0^+\right) \dfrac{(t-\alpha)^r}{r!} \end{cases} \quad (5)$$

Fractional Reduced Differential Transform Method (FRDTM) and its properties:

Within the following segment, we present the fundamental properties for the fractional reduced differential transform method (FRDTM) (Singh & Srivastava, 2015; Srivastava et al., 2014). Multiplying two independent variable functions $\pi(z)\bullet\nu(t)$ yields an analytical function called $\phi(z,t)=\pi(z)\bullet\nu(t)$. Consequently, the following is how this function may be expressed:

$$\phi(z,t)=\left(\sum_{i=0}^{\infty}\pi(i)z^{i}\right)\bullet\left(\sum_{j=0}^{\infty}\nu(j)t^{j}\right)=\sum_{k=0}^{\infty}\Phi_{k}(z)t^{k} \tag{2}$$

Definition 3.1: Spectrum function $\phi(z,t)$ of dimension 't' aimed at analytical and continuously differentiable functions is supplied in the domain with regard to space variable z and time variable t as follows (Keskin & Oturanç, 2009):

$$F_D\left[\phi(z,t)\right]=\Phi_k(z)=\frac{1}{\Gamma(\alpha k+1)}\left[\frac{\partial^{\alpha k}}{\partial t^{\alpha k}}\phi(z,t)\right]_{t=t_0} \tag{3}$$

where the time–functional derivative order is denoted by α, and $k\in\mathbb{N}\cup\{0\}$

The original function is denoted by $\phi(z,t)$, whereas the fractionally reduced altered function is denoted by $\Phi_k(z)$. To understand the inverse fractional differential transformation of $\Phi_k(z)$, let us look at the following:

$$F_D^{-1}\left[\Phi_k(z)\right]=\phi(z,t)=\sum_{k=0}^{\infty}\Phi_k(z)(t-t_0)^{\alpha k} \tag{4}$$

Combining the results of the equation (3) and the equation (4), the following is obtained:

$$\Phi_k(z)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k+1)}\left[\frac{\partial^{\alpha k}}{\partial t^{\alpha k}}\phi(z,t)\right]_{t=t_0}(t-t_0)^{\alpha k} \tag{5}$$

Whenever $t_0=0$, the equation (4) transforms into

$$\phi(z,t)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k+1)}\left[\frac{\partial^{\alpha k}}{\partial t^{\alpha k}}\phi(z,t)\right]_{t=0}t^{\alpha k} \tag{6}$$

Table 1: Axioms for the fractional reduced differential transform method (FRDTM)

Functional Form	Transformed Form
$F_D\left[\phi(z,t)\right]$	$\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}\Phi(z)\right]_{t=0}$
$F_D\left[\alpha\phi(z,t)\right]$	$\alpha\Phi(z)$ $[\alpha \text{ is constant}]$
$F_D\left[\frac{\partial^{U\alpha}}{\partial t^{U\alpha}}\phi(z,t)\right]$	$\frac{\Gamma(s\alpha+U\alpha+1)}{\Gamma(s\alpha+1)}\Phi_{s+U}(z)$
$F_D\left[\phi_1(z,t)\pm\phi_2(z,t)\right]$	$\left[\Phi_{1_k}(z)\pm\Phi_{2_k}(z)\right]$
$F_D\left[\phi_1(z,t)\times\phi_2(z,t)\right]$	$\sum_{r=0}^{k}\Phi_{1_r}(z)\Phi_{2_{k-r}}(z)$
$F_D\left[\frac{\partial^s}{\partial t^s}\phi(z,t)\right]$	$(s+1)(s+2)\ldots(s+r)\Phi_{s+r}(z)=\frac{(s+r)!}{s!}\Phi_{s+r}(z)$
$F_D\left[\frac{\partial}{\partial z}\phi(z,t)\right]$	$\frac{\partial}{\partial z}\Phi(z)$
$F_D\left[\phi(z,t)\times\frac{\partial}{\partial z}\phi(z,t)\right]$	$\sum_{r=0}^{k}[\Phi_r(z)\times(\frac{\partial\Phi_{k-r}}{\partial z})]$

An example of a wave equation (Abbasbandy, 2008) may help us understand the fundamentals of FRDTM:

$$W\phi(z,t)+A\phi(z,t)+B\phi(z,t)=h(z,t) \tag{7}$$

where following is the initial condition:

$$\phi(z,0)=s(z), \tag{8}$$

where $W=\frac{\partial}{\partial t}$ denotes the first-order linear operator, h(z,t) is the source term, B denotes the non-linear operator, and $A=\frac{\partial^2}{\partial x^2}$ denotes the second-order linear operator.

From FRDTM and Table 1, it is possible to derive the iteration formula as:

$$\frac{\Gamma\left(\alpha\left(k+1\right)+1\right)}{\Gamma\left(k\alpha+1\right)}\Phi_{k+1}(z)=H_k(z)-A\Phi_k\left(z\right)-B\Phi_k\left(z\right) \tag{9}$$

where the transformations of the functions $W\phi\left(z,t\right)$, $A\phi\left(z,t\right)$, $B\phi\left(z,t\right)$ and $h(z,t)$ are denoted by $\frac{\Gamma\left(\alpha\left(k+1\right)+1\right)}{\Gamma\left(k\alpha+1\right)}\Phi_{k+1}(z)$, $A\Phi_k\left(z\right)$, $B\Phi_k\left(z\right)$, and $H_k(z)$ respectively.

The initial condition yields,

$$\Phi_0(z)=s(z), \tag{10}$$

By substituting the equation (10) into the equation (9) and doing straightforward iterative computations, the following values of $\Phi_k\left(z\right)$ are obtained. As a result, by performing an inverse transformation on the $\left\{\Phi_k\left(z\right)\right\}_{k=0}^{n}$ values, the approximate answer is:

$$\tilde{\phi}_n(z,t)=\sum_{k=0}^{n}\Phi_k\left(z\right)t^{\alpha k} \tag{11}$$

The approximate solution’s order is n.

That is why we can get the problem’s exact solution by

$$\phi\left(z,t\right)=\lim_{n\to\infty}\tilde{\phi}_n(z,t) \tag{12}$$

Solving Fractional SIR Epidemic Model using FRDTM:

Let us rewrite the equation (1),

$$\begin{cases} {}_0^C\Delta_t^{\alpha}S(t)=-m_1S(t)I(t), \\ {}_0^C\Delta_t^{\alpha}I(t)=m_1S(t)I(t)-m_2I(t), \\ {}_0^C\Delta_t^{\alpha}R(t)=m_2I(t). \end{cases} \tag{1}$$

Also the initial conditions are taken from (Hasan et al., 2019; Kumar et al., 2020), as follows:

$S_0=620$, $I_0=10$, and $R_0=70$. $R(0)=R_0$. Also, the order of the fractional derivative Δ in reference to time ‘t’ taken through Caputo sense is $0<\alpha\le 1$, $m_1=0.001$, and $m_2=0.072$ are positive constants also taken from (Hasan et al., 2019; Kumar et al., 2020). Therefore, the equation (1) can be written as:

$$\begin{cases} {}_0^C\Delta_t^\alpha S(t) = -(0.001) S(t)\cdot I(t), \\ {}_0^C\Delta_t^\alpha I(t) = (0.001) S(t)\cdot I(t) - (0.072) I(t), \\ {}_0^C\Delta_t^\alpha R(t) = (0.072) I(t). \end{cases} \tag{13}$$

Now using the fundamental properties of FRDTM, the equation (13) can be transformed into:

$$\begin{aligned} S_{k+1}(t) &= \frac{\Gamma(k\alpha+1)}{\Gamma(\alpha(k+1)+1)}\left[-(0.001)\left\{\sum_{r=0}^{k} S_r(t) I_{k-r}(t)\right\}\right] \\ I_{k+1}(t) &= \frac{\Gamma(k\alpha+1)}{\Gamma(\alpha(k+1)+1)}\left[(0.001)\left\{\sum_{r=0}^{k} S_r(t) I_{k-r}(t)\right\} - (0.072) I_k(t)\right] \\ R_{k+1}(t) &= \frac{\Gamma(k\alpha+1)}{\Gamma(\alpha(k+1)+1)}\left[(0.072) I_k(t)\right] \end{aligned} \tag{14}$$

Similarly, the initial conditions can be transformed into:

$$\begin{cases} S(0) = \tilde{S}_0(t) = 620 \\ I(0) = \tilde{I}_0(t) = 10 \\ R(0) = \tilde{R}_0(t) = 70 \end{cases} \tag{15}$$

On replacing the equation (15) into the equation, (14) we get the successive values as follows:

$$\begin{cases} \tilde{S}_1(t) = -\frac{6.200}{\Gamma(\alpha+1)}, \tilde{S}_2(t) = -\frac{3.3356}{\Gamma(2\alpha+1)}, \\ \tilde{S}_3(t) = \frac{-1.7900888\Gamma(\alpha+1)^2 + 0.033976\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)\Gamma(\alpha+1)^2}, \dots \end{cases} \tag{16}$$

$$\begin{cases} \tilde{I}_1(t) = \frac{5.48}{\Gamma(\alpha+1)}, \tilde{I}_2(t) = \frac{2.94104}{\Gamma(2\alpha+1)}, \\ \tilde{I}_3(t) = \frac{1.57833392\Gamma(\alpha+1)^2 - 0.033976\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)\Gamma(\alpha+1)^2}, \dots \end{cases} \tag{17}$$

$$\left\{ \tilde{R}_1(t) = \frac{0.72}{\Gamma(\alpha+1)}, \tilde{R}_2(t) = \frac{0.39456}{\Gamma(2\alpha+1)}, \tilde{R}_3(t) = \frac{0.21175488}{\Gamma(3\alpha+1)}, \cdots \right. \tag{18}$$

The following four terms approximation can be gained by applying the inverse fractional reduced transformation to the equations (16), (17), and (18) respectively:

$$S_3(t)=\sum_0^3 \tilde{S}_3(t)t^{k\alpha}=\begin{cases}620-\dfrac{6.200t^{\alpha}}{\Gamma(\alpha+1)}-\dfrac{3.3356t^{2\alpha}}{\Gamma(2\alpha+1)}\\ +\left\{\dfrac{-1.7900888\Gamma(\alpha+1)^2+0.033976\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)\Gamma(\alpha+1)^2}\right\}t^{3\alpha}\end{cases} \tag{19}$$

$$I_3(t)=\sum_0^3 \tilde{I}_3(t)t^{k\alpha}=\begin{cases}10+\dfrac{5.48t^{\alpha}}{\Gamma(\alpha+1)}+\dfrac{2.94104t^{2\alpha}}{\Gamma(2\alpha+1)}\\ +\dfrac{1.57833392\Gamma(\alpha+1)^2-0.033976\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)\Gamma(\alpha+1)^2}t^{3\alpha}\end{cases} \tag{20}$$

$$R_3(t)=\sum_0^3 \tilde{R}_3(t)t^{k\alpha}=\left\{72+\dfrac{0.72t^{\alpha}}{\Gamma(\alpha+1)}+\dfrac{0.39456t^{2\alpha}}{\Gamma(2\alpha+1)}+\dfrac{0.21175488}{\Gamma(3\alpha+1)}t^{3\alpha}\right. \tag{21}$$

As described here, additional terms may be included in the series solution to increase the approach’s convergence. It is, therefore, possible to come up with an accurate solution to the problem as follows:

$$S(t)=\lim_{n\to\infty} S_n(t), I(t)=\lim_{n\to\infty} I_n(t), \text{and } R(t)=\lim_{n\to\infty} R_n(t)$$

$$S(t)=\begin{cases}620-\dfrac{6.200t^{\alpha}}{\Gamma(\alpha+1)}-\dfrac{3.3356t^{2\alpha}}{\Gamma(2\alpha+1)}\\ +\left\{\dfrac{-1.7900888\Gamma(\alpha+1)^2+0.033976\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)\Gamma(\alpha+1)^2}\right\}t^{3\alpha}+\cdots\end{cases} \tag{22}$$

$$I(t)=\begin{cases}10+\dfrac{5.48t^{\alpha}}{\Gamma(\alpha+1)}+\dfrac{2.94104t^{2\alpha}}{\Gamma(2\alpha+1)}\\ +\dfrac{1.57833392\Gamma(\alpha+1)^2-0.033976\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)\Gamma(\alpha+1)^2}t^{3\alpha}+\cdots\end{cases} \tag{23}$$

$$R(t)=\left\{72+\dfrac{0.72t^{\alpha}}{\Gamma(\alpha+1)}+\dfrac{0.39456t^{2\alpha}}{\Gamma(2\alpha+1)}+\dfrac{0.21175488}{\Gamma(3\alpha+1)}t^{3\alpha}+\cdots\right. \tag{24}$$

Using fractional reduced differential transform method (FRDTM), these polynomials could be used to approximate the solution of the fractional epidemic SIR model for $\alpha = 1$.

$$S_{15}(t)=\begin{cases}620-6.2t-1.6678t^{2}-0.2870228t^{3}-0.0337364506t^{4}\\-0.002436755662t^{5}-0.000002969792953t^{6}\\+0.00002994483993t^{7}+0.000005229584121t^{8}\\+0.0000005357913991t^{9}+0.00000002830270562t^{10}\\-0.000000001701299185t^{11}-0.0000000006438063453t^{12}\\-9.061960284\times10^{-11}t^{13}-7.767239654\times10^{-12}t^{14}\\-2.281525315\times10^{-13}t^{15}\end{cases}\quad(25)$$

$$I_{15}(t)=\begin{cases}10+5.48t+1.47052t^{2}+0.25173032t^{3}+0.02920530484t^{4}\\+0.002016199272t^{5}-0.00002122459831t^{6}\\-0.00002972652978t^{7}-0.000004962045353t^{8}\\-0.0000004960950363t^{9}-0.00000002473082136t^{10}\\+0.000000001863173652t^{11}+0.0000000006326273034t^{12}\\+8.711582085\times10^{-11}t^{13}+7.319215433\times10^{-12}t^{14}\\+1.930202974\times10^{-13}t^{15}\end{cases}\quad(26)$$

$$R_{15}(t)=\begin{cases}70+0.72t+0.19728t^{2}+0.03529248t^{3}+0.004531145760t^{4}\\+0.0004205563897t^{5}+0.00002419439126t^{6}\\-0.0000002183101541t^{7}-0.0000002675387680t^{8}\\-0.00000003969636282t^{9}-0.000000003571884261t^{10}\\-0.0000000001618744671t^{11}+1.117904191\times10^{-11}t^{12}\\+3.503781988\times10^{-12}t^{13}+4.480242215\times10^{-13}t^{14}\\+3.513223408\times10^{-14}t^{15}\end{cases}\quad(27)$$

Results and Discussion:

Absolute error and relative error have been calculated among fractional reduced differential transform method (FRDTM) and residual power series method (RPSM), where the definitions of these terms are as follows:

$$\text{Absolute error}=\left|\text{FRDTM}_{\text{output}}-\text{RPSM}_{\text{output}}\right| \quad (28)$$

and

$$\text{Relative error}=\left|\frac{\text{FRDTM}_{\text{output}}-\text{RPSM}_{\text{output}}}{\text{RPSM}_{\text{output}}}\right| \quad (29)$$

Figures 1–3 clearly demonstrate the excellent conformity among the fractional reduced differential transform method (FRDTM) & residual power series method (RPSM) based on the answers produced. Absolute error and relative error as defined above are plotted in 2D graphs for susceptible-S(t), infected-I(t), and recovered-R(t) values in figures 4-6 and figures 7-9 respectively, also both errors' numerical values are shown in the tables 2, 3, and 4. From these figures 4-9 and tables 2-4, we can clearly see that error is negligible, so both FRDTM and RPSM agree with each other really well. Using Figures 10–12, we've shown the 3D plots of the susceptible-S(t), infected-I(t), and recovered-R(t) individuals using the fractional reduced differential transform method (FRDTM). Since derivative having order in fraction allows for more flexibility than an integer order one, we can find solutions for any number of orders of derivatives using the suggested technique FRDTM (see Figures 13–15). Figure 16 shows the population versus time graph of the FRDTM solution. Tables 2, 3, and 4 present a numerical comparison between the 15th-FRDTM solution and the 15th-RPSM solution for $\alpha = 1$ in order to demonstrate the reliability for the fractional reduced differential transform method (FRDTM) aimed at estimating solution for proposed SIR epidemic model.

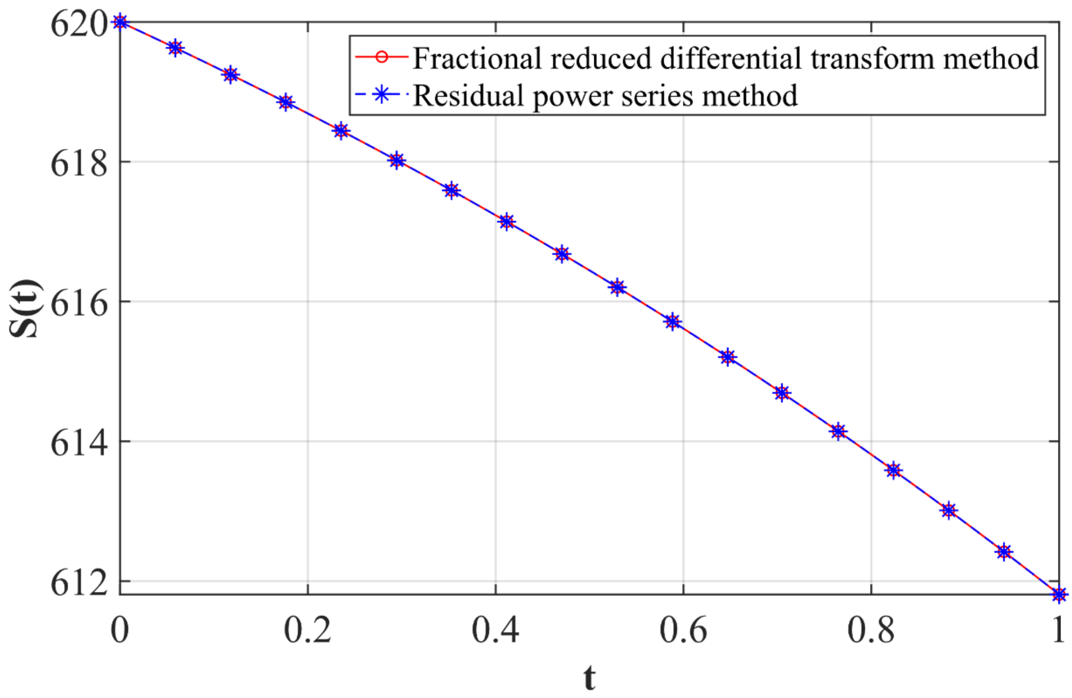

Figure 1: FRDTM and RPSM 2D-plot of susceptible persons S(t) w.r.t time t at $\alpha = 1$.

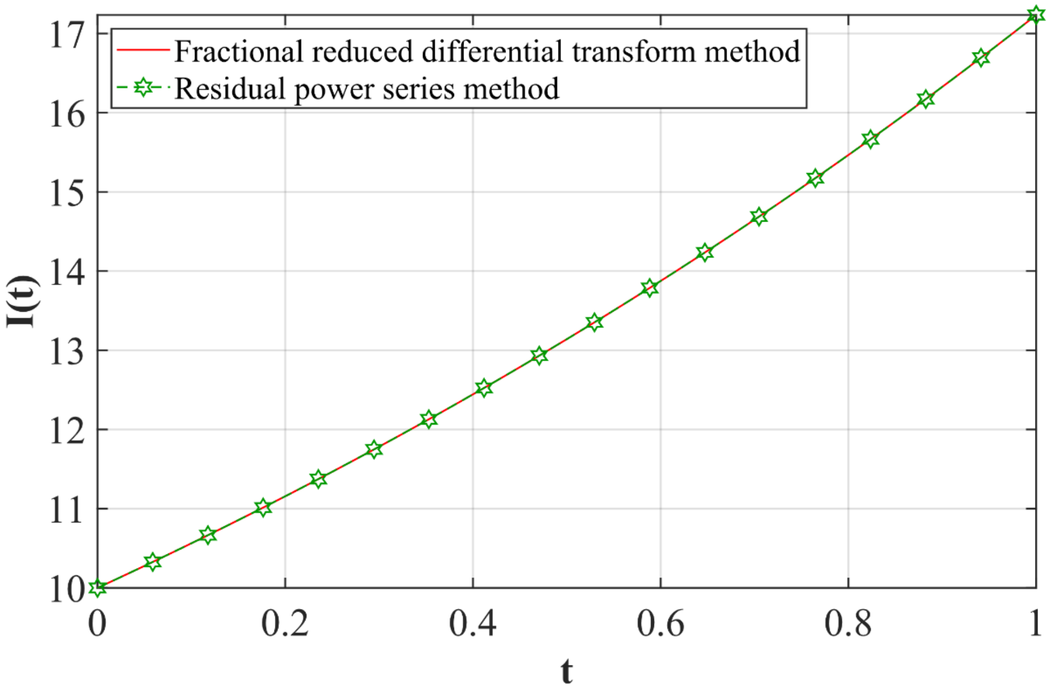

Figure 2: FRDTM and RPSM 2D-plot of infected persons I(t) w.r.t time t at $\alpha = 1$.

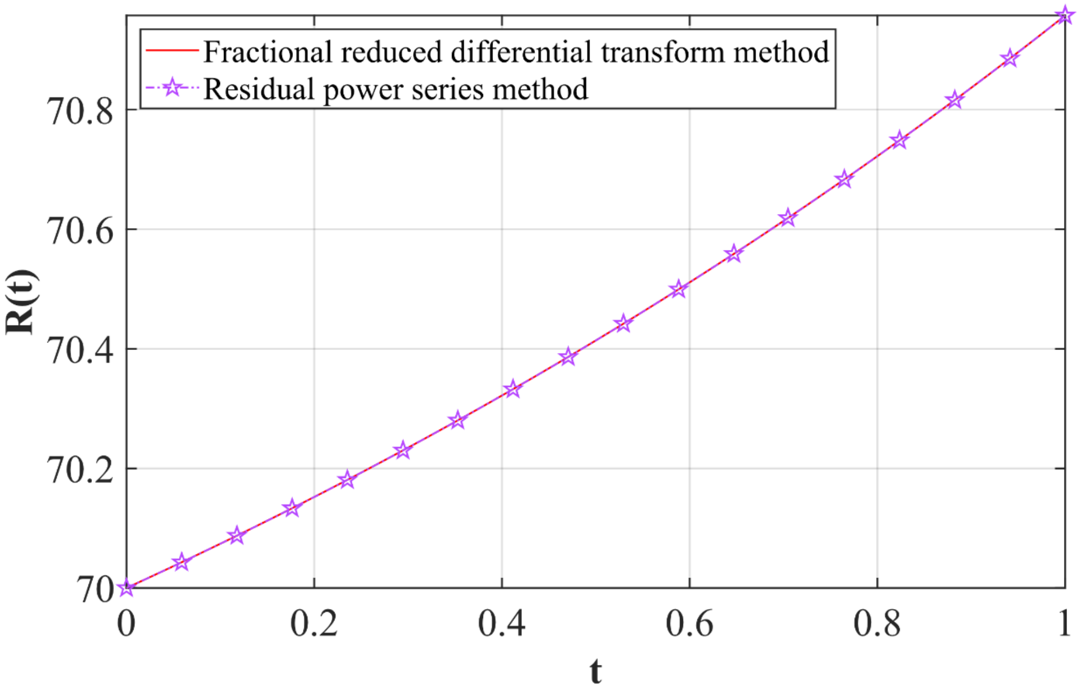

Figure 3: FRDTM and RPSM 2D-plot of recovered persons R(t) w.r.t time t at $\alpha = 1$.

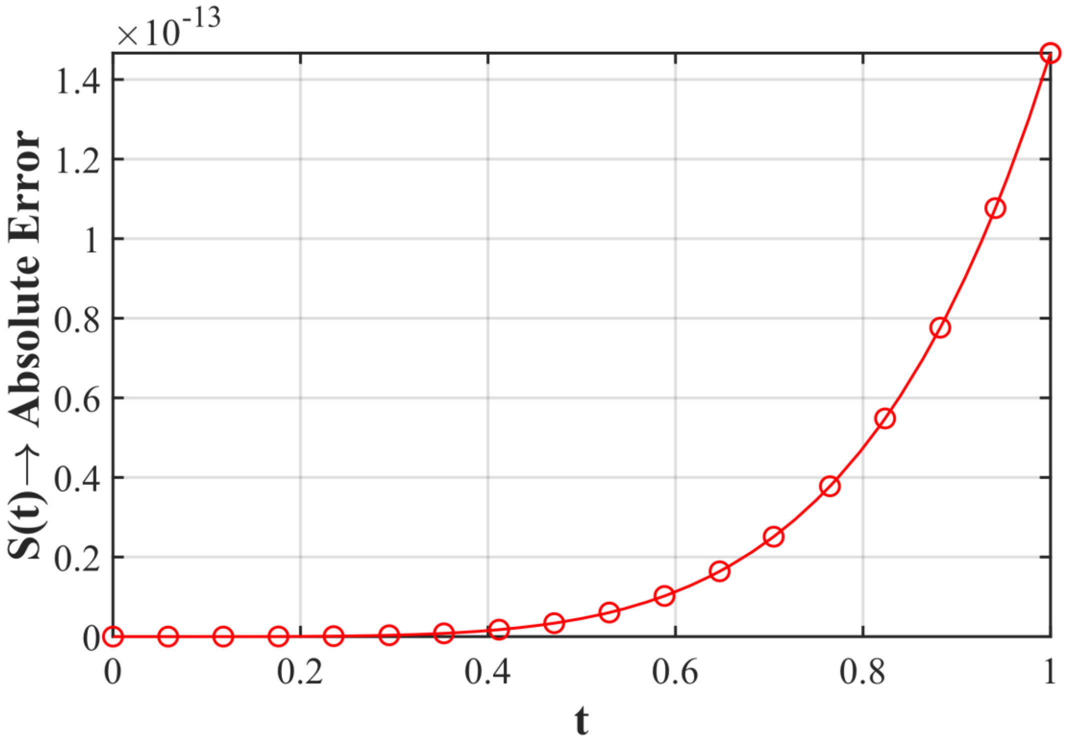

Figure 4: At $\alpha = 1$, the absolute error of susceptible persons S(t) versus time t plot using FRDTM and RPSM outputs.

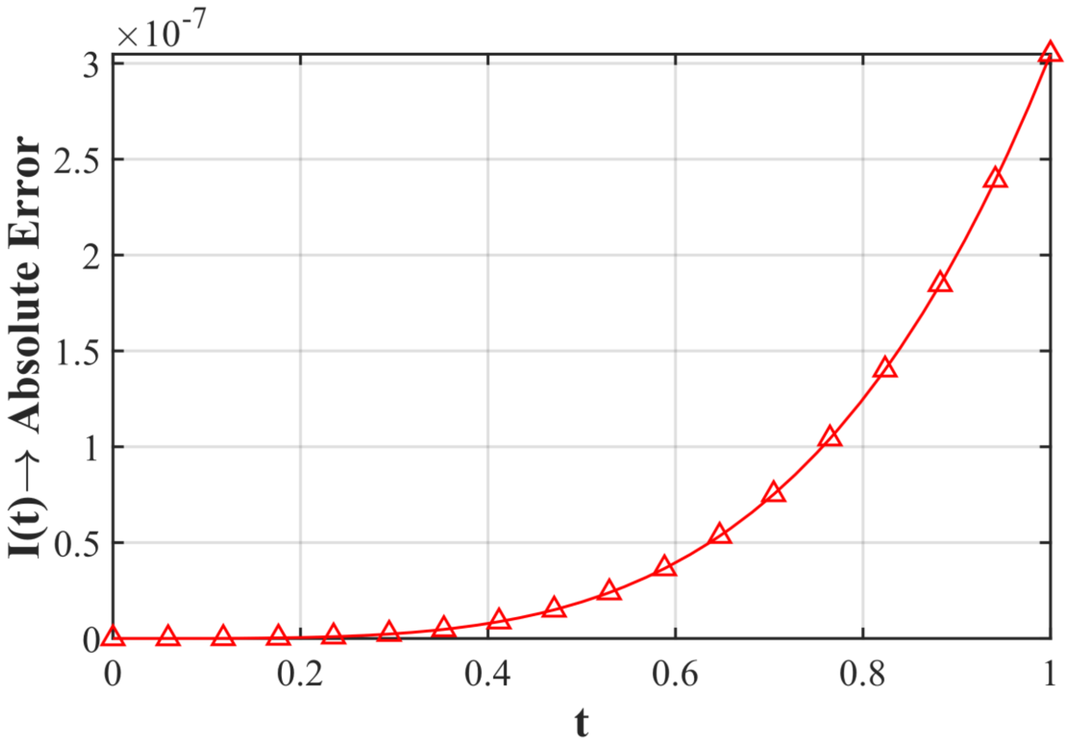

Figure 5: At $\alpha = 1$, the absolute error of infected persons I(t) versus time t plot using FRDTM and RPSM outputs.

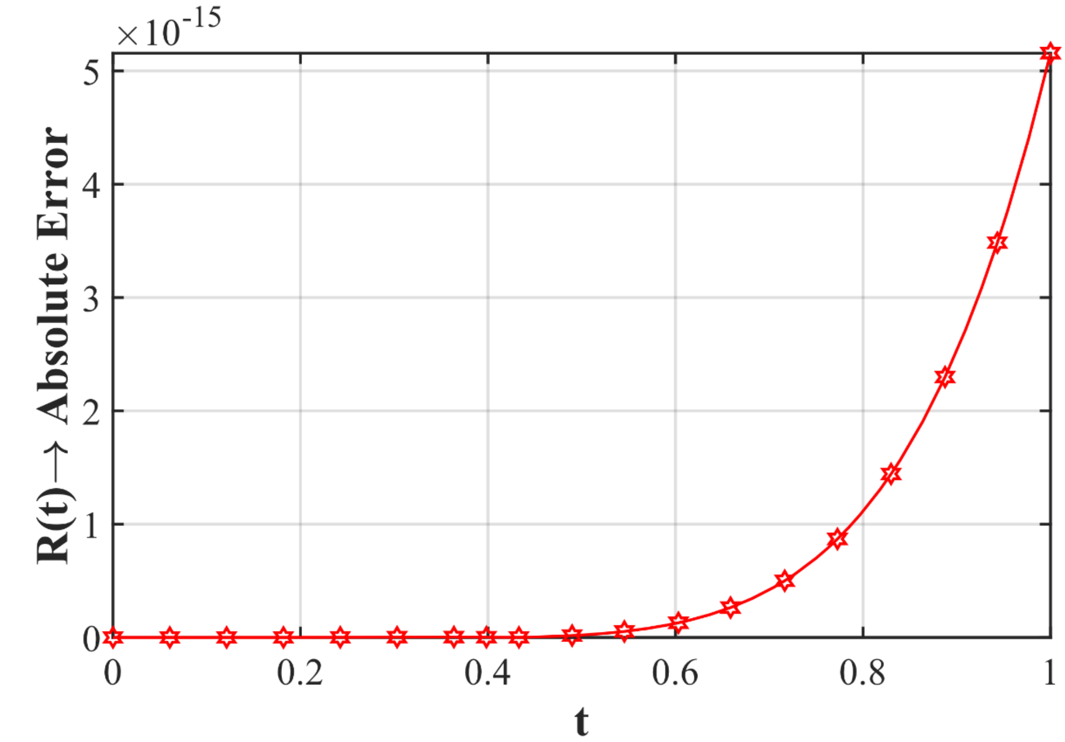

Figure 6: At $\alpha = 1$, the absolute error of recovered persons R(t) versus time t plot using FRDTM and RPSM outputs.

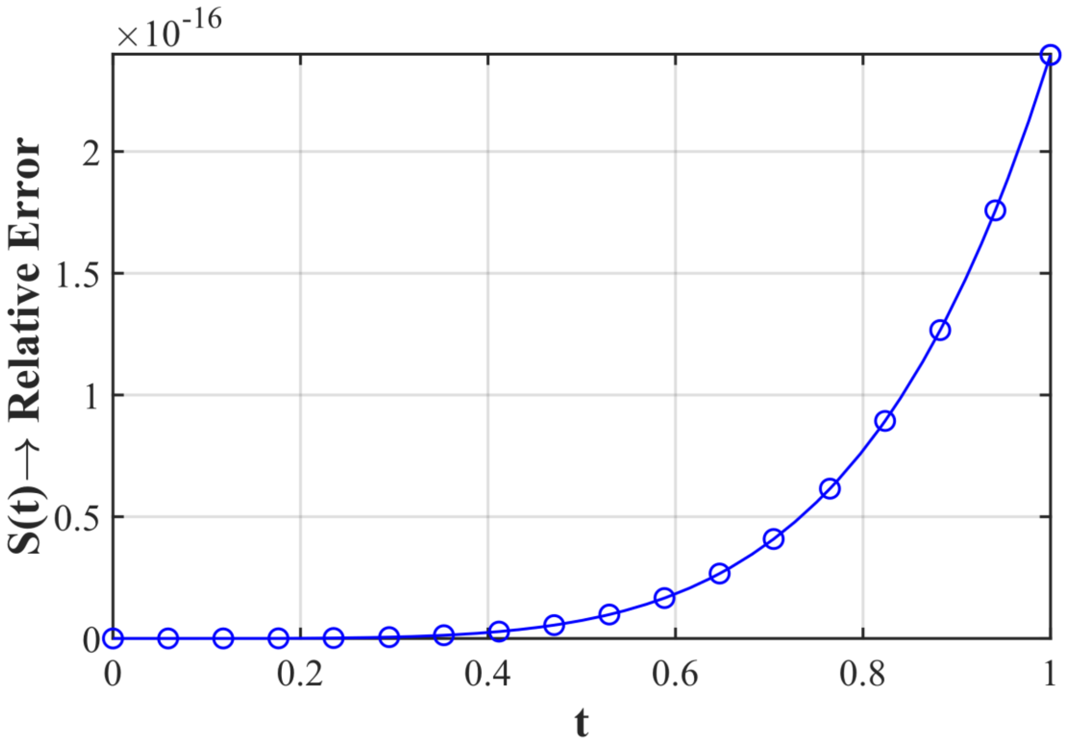

Figure 7: At $\alpha = 1$, the relative error of susceptible persons S(t) versus time t plot using FRDTM and RPSM outputs.

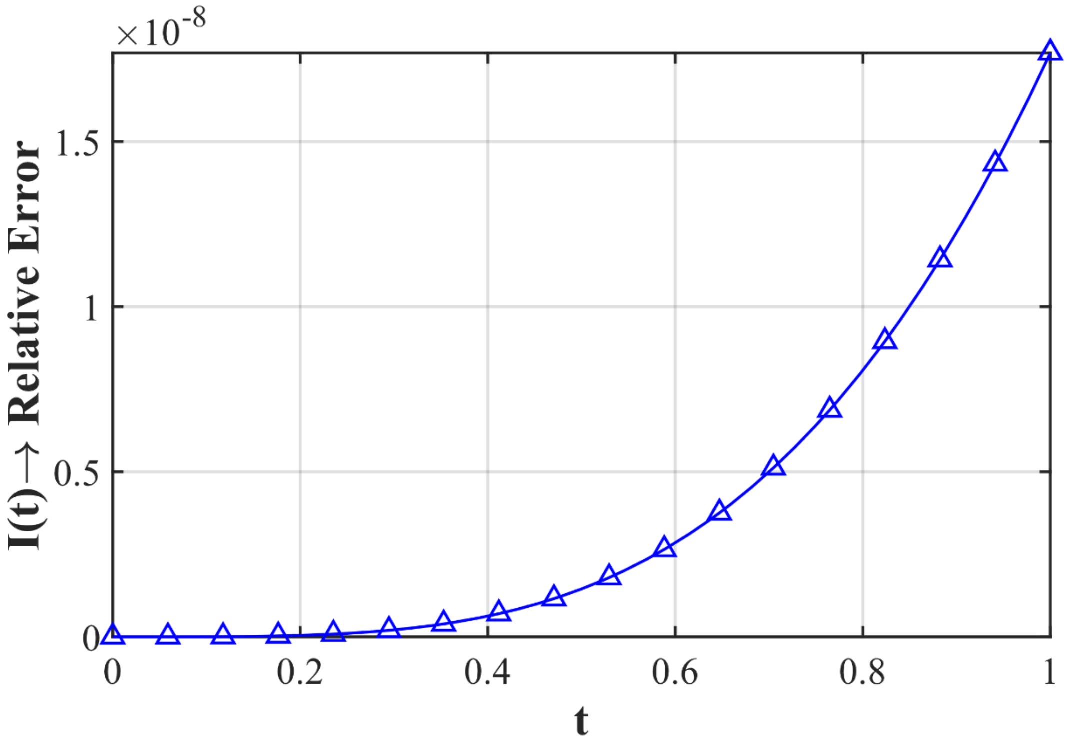

Figure 8: At $\alpha = 1$, the relative error of infected persons I(t) versus time t plot using FRDTM and RPSM outputs.

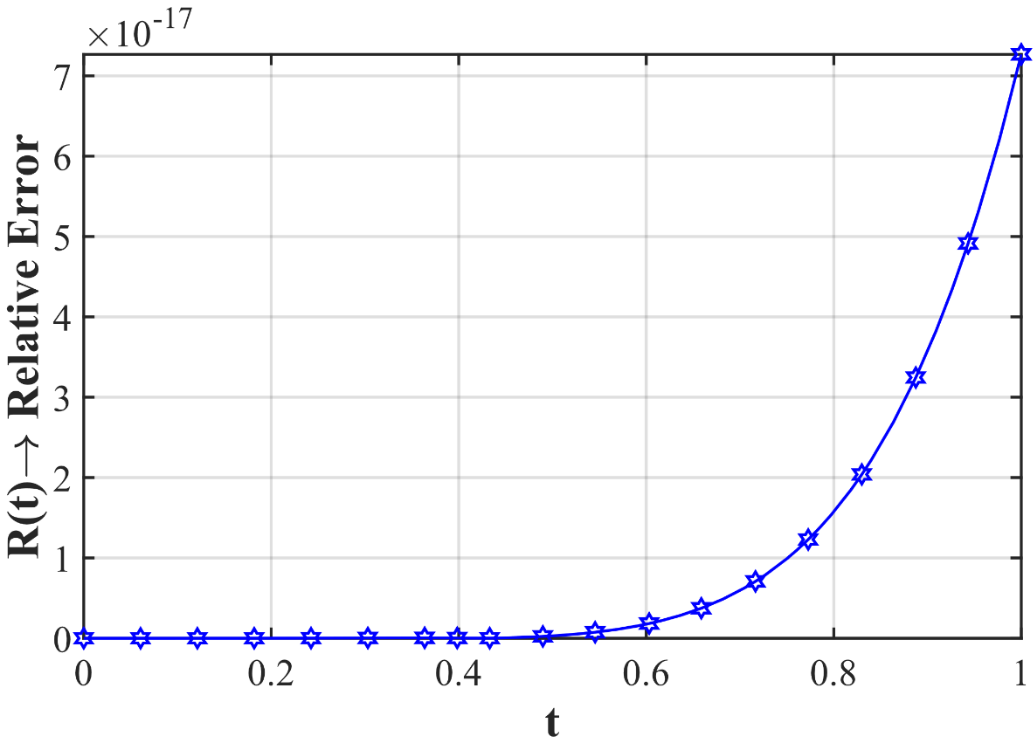

Figure 9: At $\alpha = 1$, the relative error of recovered persons R(t) versus time t plot using FRDTM and RPSM outputs.

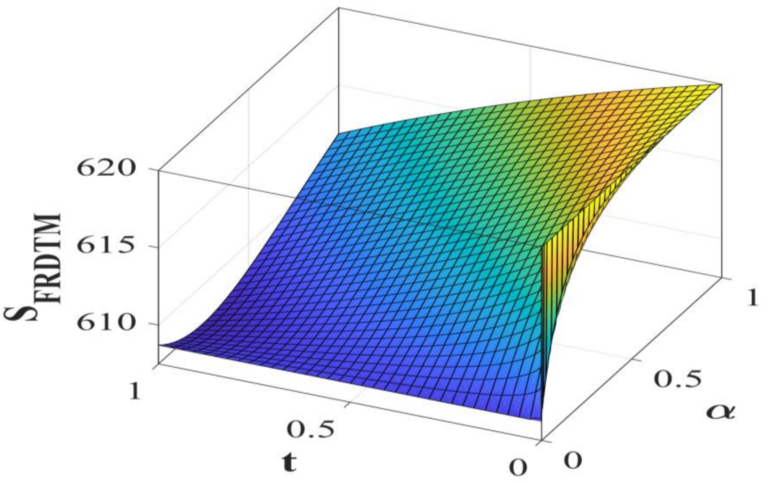

Figure 10: 3D plot of susceptible people's behaviour in reference with time t & $0 < \alpha < 1$ by FRDTM.

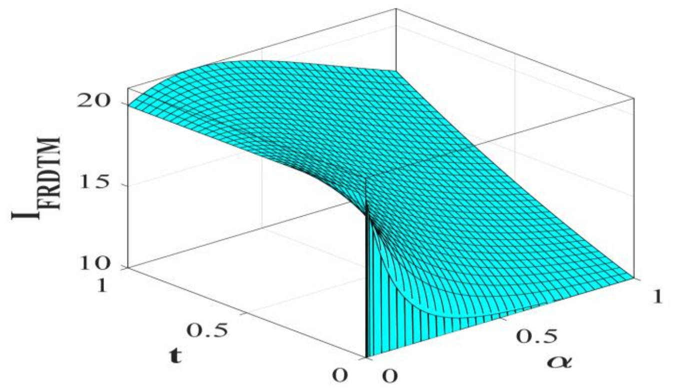

Figure 11: 3D plot of infected people's behaviour in reference with time t & $0 < \alpha < 1$ by FRDTM.

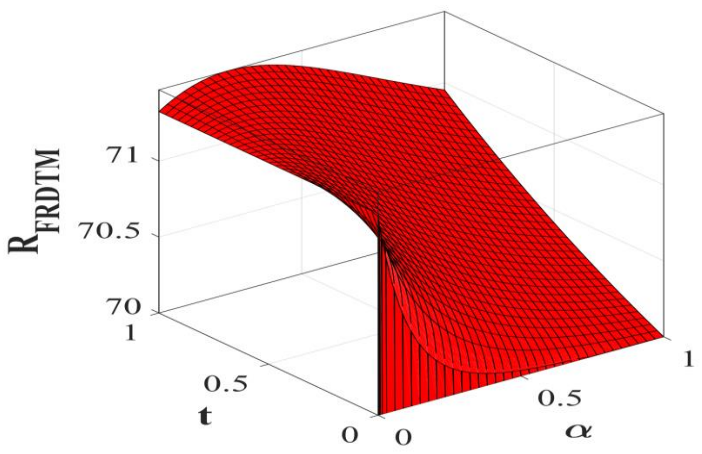

Figure 12: 3D plot of recovered people's behaviour in reference with time t & $0 < \alpha < 1$ by FRDTM.

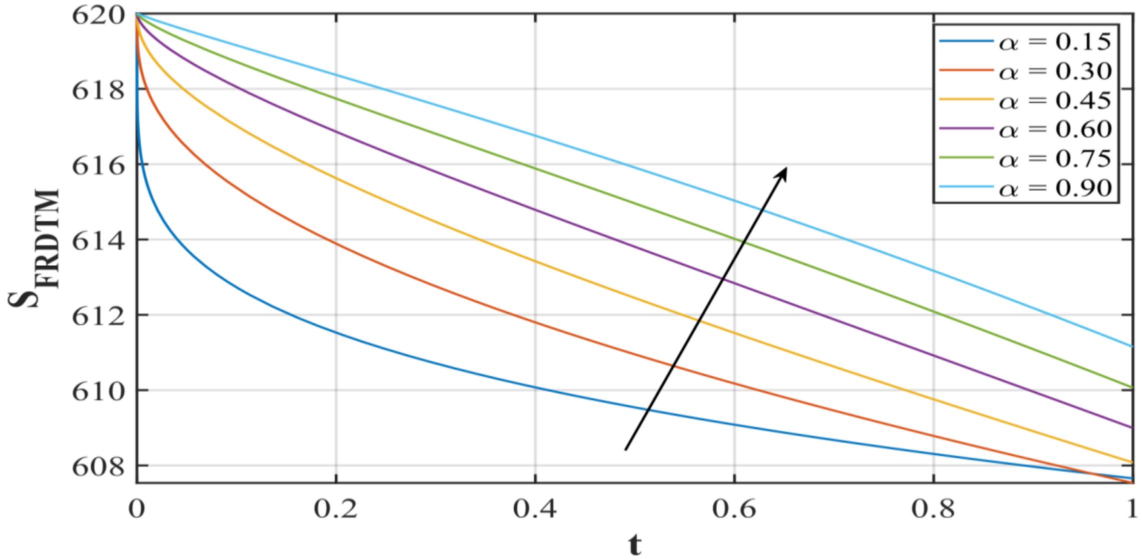

Figure 13: Susceptible people's behaviour in reference with time t & for distinct values of α by FRDTM.

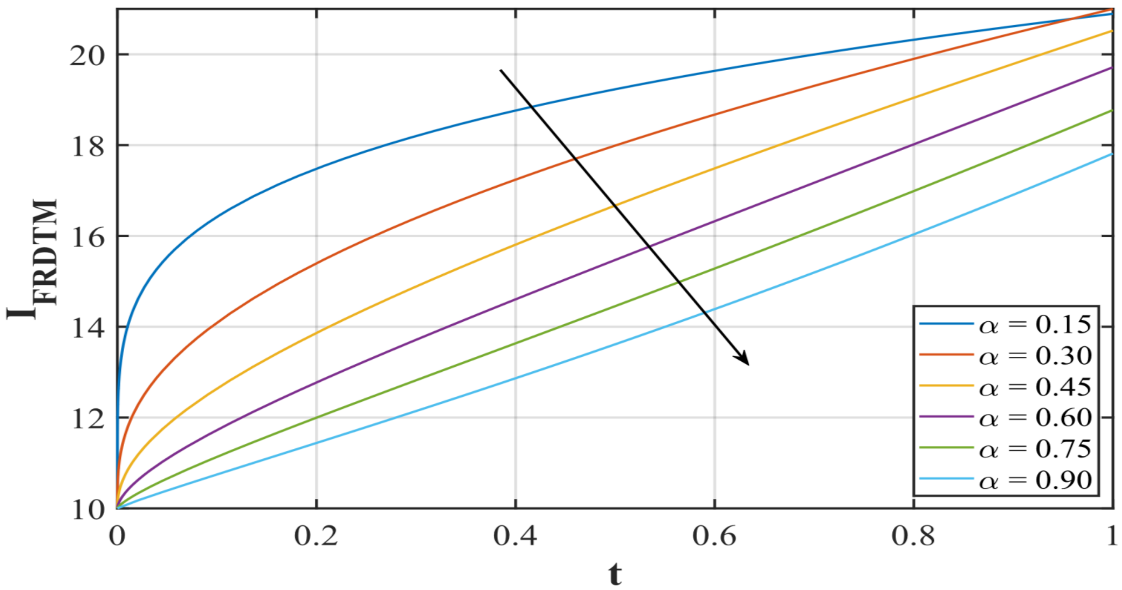

Figure 14: Infected people's behaviour in reference with time t & for distinct values of α by FRDTM.

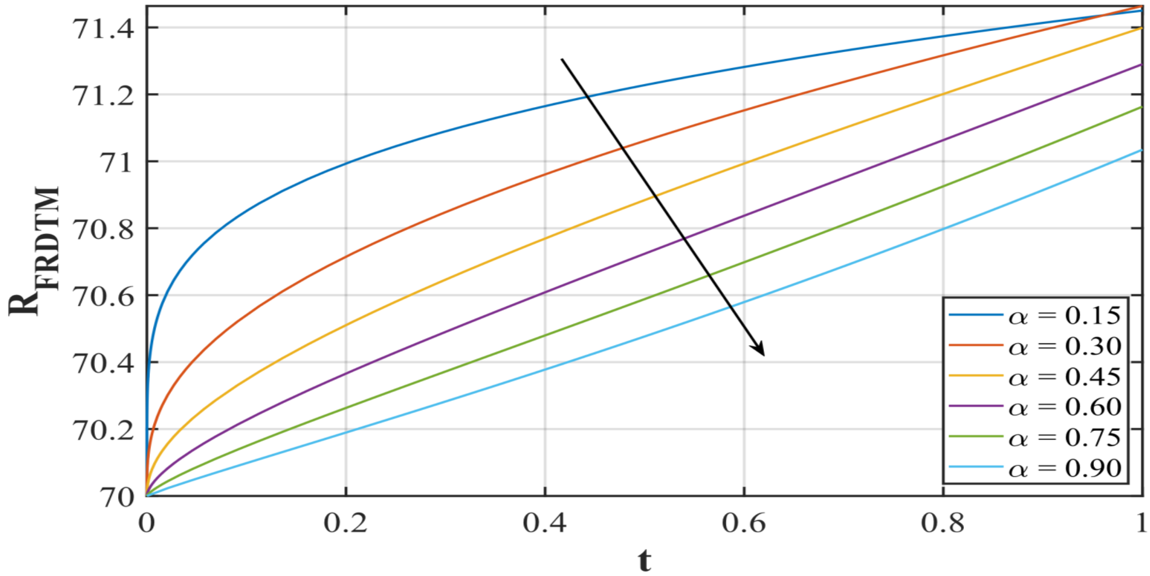

Figure 15: Recovered people's behaviour in reference with time t & for distinct values of α by FRDTM.

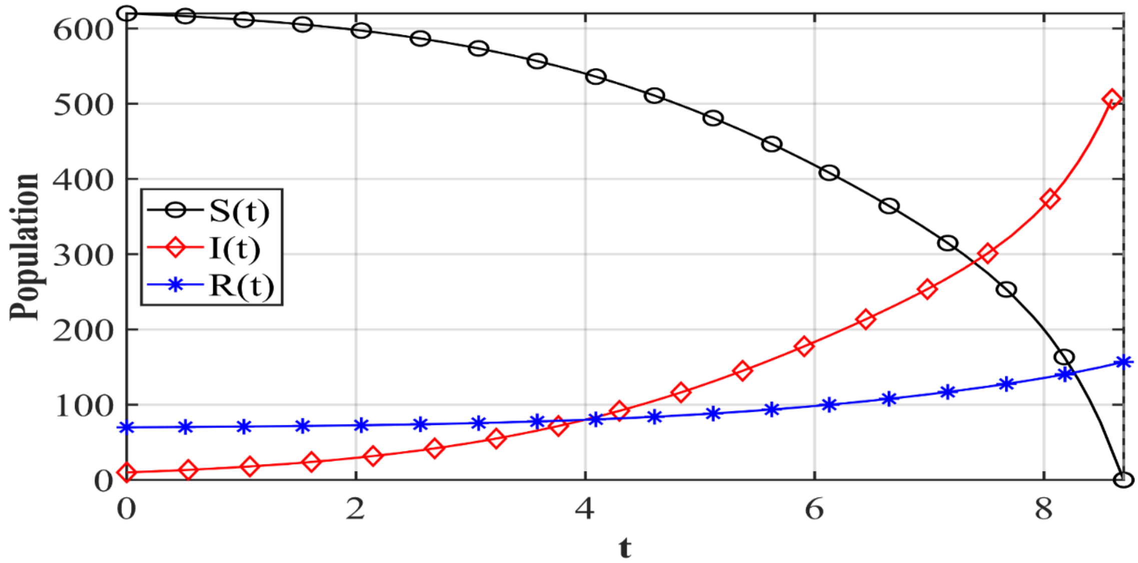

Figure 16: 2D plot for population versus time t using FRDTM. $(0 < t < 8.695)$

Table 2: Solution of S(t) by using FRDTM and RPSM (Hasan et al., 2019).

t	S(t) by FRDTM, $\alpha=1$	S(t) by RPSM, $\alpha=1$	Absolute Error in S(t)	Relative Error in S(t)
0	620	620	0	0
0.15	619.0314885	619.0314885	$4.385353335\times10^{-19}$	$7.084216902\times10^{-22}$
0.30	617.9818692	617.9818692	$2.478414995\times10^{-17}$	$4.010497910\times10^{-20}$
0.45	616.8446872	616.8446872	$2.448922481\times10^{-16}$	$3.970079552\times10^{-19}$
0.60	615.6130342	615.6130342	$1.165513788\times10^{-15}$	$1.893257164\times10^{-18}$
0.75	614.2795261	614.2795261	$3.642186357\times10^{-15}$	$5.929200307\times10^{-18}$
0.90	612.8362842	612.8362842	8.47136660×10^{-15}	$1.382321318\times10^{-17}$

Table 3: Solution of I(t) by using FRDTM and RPSM (Hasan et al., 2019).

t	I(t) by FRDTM, $\alpha=1$	I(t) by RPSM, $\alpha=1$	Absolute Error in I(t)	Relative Error in I(t)
0	10	10	0	0
0.15	10.85595123	10.85595123	$1.543231243\times10^{-10}$	$1.421553220\times10^{-11}$
0.30	11.78338495	11.78338495	2.469135997×10^{-9}	$2.095438624\times10^{-10}$
0.45	12.78795373	12.78795372	1.249982902×10^{-8}	$9.774690527\times10^{-10}$
0.60	13.87570084	13.87570080	3.950508955×10^{-8}	2.847069855×10^{-9}
0.75	15.05307713	15.05307703	9.644664893×10^{-8}	6.407105206×10^{-9}
0.90	16.32695691	16.32695671	1.999890321×10^{-7}	1.224900838×10^{-8}

Table 4: Solution of R(t) by using FRDTM and RPSM (Hasan et al., 2019).

t	R(t) by FRDTM, $\alpha = 1$	R(t) by RPSM, $\alpha = 1$	Absolute Error in R(t)	Relative Error in R(t)
0	70	70	0	0
0.15	70.11256023	70.11256023	$1.130771978 \times 10^{-19}$	$1.612795161 \times 10^{-21}$
0.30	70.23474584	70.23474584	$7.186997627 \times 10^{-18}$	$1.023282357 \times 10^{-19}$
0.45	70.36735900	70.36735900	$8.132732917 \times 10^{-17}$	$1.155753610 \times 10^{-18}$
0.60	70.51126504	70.51126504	$4.540565084 \times 10^{-16}$	$6.439488900 \times 10^{-18}$
0.75	70.66739675	70.66739675	$1.721028449 \times 10^{-15}$	$2.435392456 \times 10^{-17}$
0.90	70.83675885	70.83675885	$5.102917432 \times 10^{-15}$	$7.203770353 \times 10^{-17}$

Conclusion:

In this research paper for the non-linear SIR epidemic model with fractional order, we use a novel application of the fractional reduced differential transform (FRDTM). Also, here, it is shown outputs for the fractional reduced differential transform (FRDTM) as two-dimensional & three-dimensional plots. The fractional reduced differential transform (FRDTM) solution is in the form of a convergent power series with beautifully calculated components that do not need linearization, perturbation, or discretization. Traditional techniques need more time and effort to implement; thus, this approach is less time-consuming than other methods. In this research, error analysis is done numerically and graphically among the fractional reduced differential transform method (FRDTM) & residual power series method (RPSM). By comparing the suggested approach's findings to the residual power series method (RPSM), we found that the fractional reduced differential transform method (FRDTM) yielded more accurate approximations. Fractional SIR epidemic models for various values of α were solved using the fractional reduced differential transform method (FRDTM) for determining how fractional derivative affected them. We found that the curves for fractional SIR epidemic model solutions resembled those of classical SIR epidemic model solutions.

Funding: This study is not supported by any other sources for funding.

Conflicts of Interest: The authors disclose that they have no competing interests.

References:

I. Abbasbandy, S. (2008). Numerical method for non-linear wave and diffusion equations by the variational iteration method. International Journal for Numerical Methods in Engineering, 73(12), 1836–1843. https://doi.org/10.1002/nme.2150

II. Abuteen, E., Freihat, A., Al-Smadi, M., Khalil, H., & Khan, R. A. (2016). Approximate series solution of nonlinear, fractional Klein-Gordon equations using fractional reduced differential transform method. Journal of Mathematics and Statistics, 12(1), 23–33. https://doi.org/10.3844/jmssp.2016.23.33

III. Allen, L. J. S. (1994). Some discrete-time SI, SIR, and SIS epidemic models. Mathematical Biosciences, 124(1), 83–105. https://doi.org/10.1016/0025-5564(94)90025-6

IV. Ameen, I., & Novati, P. (2017). The solution of fractional order epidemic model by implicit Adams methods. Applied Mathematical Modelling, 43, 78–84. https://doi.org/10.1016/j.apm.2016.10.054

V. Arqub, O. A., & El-Ajou, A. (2013). Solution of the fractional epidemic model by homotopy analysis method. Journal of King Saud University - Science, 25(1), 73–81. https://doi.org/10.1016/j.jksus.2012.01.003

VI. Balint, A. M., Balint, S., & Neculae, A. (2022). On the objectivity of mathematical description of ion transport processes using general temporal Caputo and Riemann-Liouville fractional partial derivatives. Chaos, Solitons and Fractals, 156, 111802. https://doi.org/10.1016/j.chaos.2022.111802

VII. Banatvala, J. E., & Brown, D. W. G. (2004). Rubella. Lancet, 363(9415), 1127–1137. https://doi.org/10.1016/S0140-6736(04)15897-2

VIII. Becker, N. (1979). The Uses of Epidemic Models. Biometrics, 35(1), 295. https://doi.org/10.2307/2529951

IX. Ciotti, M., Ciccozzi, M., Terrinoni, A., Jiang, W. C., Wang, C. Bin, & Bernardini, S. (2020). The COVID-19 pandemic. Critical Reviews in Clinical Laboratory Sciences, 57(6), 365–388. https://doi.org/10.1080/10408363.2020.1783198

X. Din, A., Khan, F. M., Khan, Z. U., Yusuf, A., & Munir, T. (2022). The mathematical study of climate change model under nonlocal fractional derivative. Partial Differential Equations in Applied Mathematics, 5, 100204. https://doi.org/10.1016/j.padiff.2021.100204

XI. Ebaid, A., & Al-Jeaid, H. K. (2022). The Mittag–Leffler Functions for a Class of First-Order Fractional Initial Value Problems: Dual Solution via Riemann–Liouville Fractional Derivative. Fractal and Fractional, 6(2), 85. https://doi.org/10.3390/fractalfract6020085

XII. Grassly, N. C., & Fraser, C. (2008). Mathematical models of infectious disease transmission. Nature Reviews Microbiology, 6(6), 477–487. https://doi.org/10.1038/nrmicro1845

XIII. Grossman, Z., Meier-Schellersheim, M., Paul, W. E., & Picker, L. J. (2006). Pathogenesis of HIV infection: What the virus spares is as important as what it destroys. Nature Medicine, 12(3), 289–295. https://doi.org/10.1038/nm1380

XIV. Gupta, P. K. (2011). Approximate analytical solutions of fractional BenneyLin equation by reduced differential transform method and the homotopy perturbation method. Computers and Mathematics with Applications, 61(9), 2829–2842. https://doi.org/10.1016/j.camwa.2011.03.057

XV. Haq, F., Shahzad, M., Muhammad, S., Wahab, H. A., & Ur Rahman, G. (2017). Numerical Analysis of Fractional Order Epidemic Model of Childhood Diseases. Discrete Dynamics in Nature and Society, 2017, 1–7. https://doi.org/10.1155/2017/4057089

XVI. Hasan, S., Al-Zoubi, A., Freihet, A., Al-Smadi, M., & Momani, S. (2019). Solution of fractional SIR epidemic model using residual power series method. Applied Mathematics and Information Sciences, 13(2), 153–161. https://doi.org/10.18576/AMIS/130202

XVII. Hethcote, H. W. (1994). A Thousand and One Epidemic Models (pp. 504–515). https://doi.org/10.1007/978-3-642-50124-1_29

XVIII. Keskin, Y., & Oturanç, G. (2009). Reduced differential transform method for partial differential equations. International Journal of Nonlinear Sciences and Numerical Simulation, 10(6), 741–749. https://doi.org/10.1515/IJNSNS.2009.10.6.741

XIX. Kumar, S., Ahmadian, A., Kumar, R., Kumar, D., Singh, J., Baleanu, D., & Salimi, M. (2020). An efficient numerical method for fractional SIR epidemic model of infectious disease by using bernstein wavelets. Mathematics, 8(4), 558. https://doi.org/10.3390/math8040558

XX. Lan, K. (2022). Linear first order Riemann-Liouville fractional differential and perturbed Abel's integral equations. Journal of Differential Equations, 306, 28–59. https://doi.org/10.1016/j.jde.2021.10.025

XXI. Liang, J., Mu, Y., & Xiao, T. J. (2022). Nonlocal integro-differential equations of Sobolev type in Banach spaces involving ψ -Caputo fractional derivative. Banach Journal of Mathematical Analysis, 16(1), 3. https://doi.org/10.1007/s43037-021-00155-5

XXII. Luchko, Y. (2022). Fractional Differential Equations with the General Fractional Derivatives of Arbitrary Order in the Riemann–Liouville Sense. Mathematics, 10(6). https://doi.org/10.3390/math10060849

XXIII. Patel, H. S., & Tandel, P. V. (2021). Fractional Reduced Differential Transform Method for the Water Transport in Unsaturated Porous Media. In International Journal of Applied and Computational Mathematics (Vol. 7, Issue 1, pp. 1–14). Springer. https://doi.org/10.1007/s40819-020-00940-0

XXIV. Perry, R. T., & Halsey, N. A. (2004). The clinical significance of measles: A review. Journal of Infectious Diseases, 189(SUPPL. 1), S4–S16. https://doi.org/10.1086/377712

XXV. Poynard, T., Yuen, M. F., Ratziu, V., & Lung Lai, C. (2003). Viral hepatitis C. Lancet, 362(9401), 2095–2100. https://doi.org/10.1016/S0140-6736(03)15109-4

XXVI. Rawashdeh, M. S. (2017). A reliable method for the space-time fractional Burgers and time-fractional Cahn-Allen equations via the FRDTM. Advances in Difference Equations, 2017(1), 99. https://doi.org/10.1186/s13662-017-1148-8

XXVII. Ruziev, M., & Zunnunov, R. (2022). On a Nonlocal Problem for Mixed-Type Equation with Partial Riemann-Liouville Fractional Derivative. Fractal and Fractional, 6(2), 110. https://doi.org/10.3390/fractalfract6020110

XXVIII. Sarin, S. K., & Kumar, M. (2012). Hepatitis E. Zakim and Boyer's Hepatology, 367(13), 605–628. https://doi.org/10.1016/B978-1-4377-0881-3.00033-4

XXIX. Sene, N. (2020). SIR epidemic model with Mittag–Leffler fractional derivative. Chaos, Solitons and Fractals, 137, 109833. https://doi.org/10.1016/j.chaos.2020.109833

XXX. Sene, N. (2022). Analytical Solutions of a Class of Fluids Models with the Caputo Fractional Derivative. Fractal and Fractional, 6(1), 35. https://doi.org/10.3390/fractalfract6010035

XXXI. Singh, B. K., & Kumar, P. (2018). FRDTM for numerical simulation of multi-dimensional, time-fractional model of Navier–Stokes equation. Ain Shams Engineering Journal, 9(4), 827–834. https://doi.org/10.1016/j.asej.2016.04.009

XXXII. Singh, B. K., & Srivastava, V. K. (2015). Approximate series solution of multi-dimensional, time fractional-order (heat-like) diffusion equations using FRDTM. Royal Society Open Science, 2(4), 140511. https://doi.org/10.1098/rsos.140511

XXXIII. Srivastava, V. K., Kumar, S., Awasthi, M. K., & Singh, B. K. (2014). Two-dimensional time fractional-order biological population model and its analytical solution. Egyptian Journal of Basic and Applied Sciences, 1(1), 71–76. https://doi.org/10.1016/j.ejbas.2014.03.001

XXXIV. Tamboli, V. K., & Tandel, P. V. (2022). Solution of the time-fractional generalized Burger–Fisher equation using the fractional reduced differential transform method. Journal of Ocean Engineering and Science, 7(4), 399–407. https://doi.org/10.1016/j.joes.2021.09.009

XXXV. Tandel, P., Patel, H., & Patel, T. (2022). Tsunami wave propagation model: A fractional approach. Journal of Ocean Engineering and Science, 7(6), 509–520. https://doi.org/10.1016/j.joes.2021.10.004

XXXVI. Tiollais, P., Pourcel, C., & Dejean, A. (1985). The hepatitis B virus. Nature, 317(6037), 489–495. https://doi.org/10.1038/317489a0

XXXVII. Vellappandi, M., Kumar, P., Govindaraj, V., & Albalawi, W. (2022). An optimal control problem for mosaic disease via Caputo fractional derivative. Alexandria Engineering Journal, 61(10), 8027–8037. https://doi.org/10.1016/j.aej.2022.01.055

XXXVIII. William Ogilvy Kermack, A. G. M. (1927). A contribution to the mathematical theory of epidemics. Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character, 115(772), 700–721. https://doi.org/10.1098/rspa.1927.0118

XXXIX. Wimmer, E., Hellen, C. U. T., & Cao, X. (1993). Genetics of poliovirus. Annual Review of Genetics, 27(1), 353–436. https://doi.org/10.1146/annurev.ge.27.120193.002033

XL. Yildirim, A., & Cherruault, Y. (2009). Analytical approximate solution of a SIR epidemic model with constant vaccination strategy by homotopy perturbation method. Kybernetes, 38(9), 1566–1575. https://doi.org/10.1108/03684920910991540

XLI. Zhou, J. (1986). Differential Transformation and Its Applications for Electronic Circuits, Huazhong Science & Technology University Press, China. Huazhong University Press, Wuhan, China.